

Decompositions of Besov-Hausdorff and Triebel-Lizorkin-Hausdorff Spaces and Their Applications

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Abstract Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, 1 - \frac{1}{\max\{p, q\}}]$. In this paper, the authors establish the φ -transform characterizations of Besov-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ ($q > 1$); as applications, the authors then establish their embedding properties (which on $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is also sharp), smooth atomic and molecular decomposition characterizations for suitable τ . Moreover, using their atomic and molecular decomposition characterizations, the authors investigate the trace properties and the boundedness of pseudo-differential operators with homogeneous symbols in $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ ($q > 1$), which generalize the corresponding classical results on homogeneous Besov and Triebel-Lizorkin spaces when $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$.

1 Introduction

To establish the connections between Besov and Triebel-Lizorkin spaces with Q spaces, which was an open problem proposed by Dafni and Xiao in [6], Yang and Yuan [30, 31] introduced new classes of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which unify and generalize the Besov spaces $\dot{B}_{p,q}^s(\mathbb{R}^n)$, Triebel-Lizorkin spaces $\dot{F}_{p,q}^s(\mathbb{R}^n)$, Morrey spaces, Morrey-Triebel-Lizorkin spaces and Q spaces. We pointed out that the Q spaces on \mathbb{R}^n were originally introduced by Essén, Janson, Peng and Xiao [8]; see also [6, 8, 27, 28] for the history of Q spaces and their properties.

Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, \frac{1}{(\max\{p, q\})^\tau}]$, where and in what follows, t' denotes the conjugate index of $t \in [1, \infty)$. The Besov-Hausdorff spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-Hausdorff spaces $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ ($q > 1$) were also introduced in [30, 31]; moreover, it was proved therein that they are, respectively, the predual spaces of $\dot{B}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$. The spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were originally called the Hardy-Hausdorff spaces in [30, 31]. However, it seems that it is more reasonable to call them, respectively, the Besov-Hausdorff spaces and the Triebel-Lizorkin-Hausdorff spaces. The spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ unify and generalize the Besov space $\dot{B}_{p,q}^s(\mathbb{R}^n)$,

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the Triebel-Lizorkin space $\dot{F}_{p,q}^s(\mathbb{R}^n)$ and the Hardy-Hausdorff space $HH_{-\alpha}^1(\mathbb{R}^n)$, where $HH_{-\alpha}^1(\mathbb{R}^n)$ was introduced in [6] and was proved to be the predual space of the space $Q_\alpha(\mathbb{R}^n)$ therein.

It is well known that the wavelet decomposition plays an important role in the study of function spaces and their applications; see, for example, [19, 20] and their references. Moreover, the φ -transform decomposition of Frazier and Jawerth [10, 11, 12] is very similar in spirit to the wavelet decomposition, which is also proved to be a powerful tool in the study of function spaces and boundedness of operators, and was further developed by Bownik [3, 4]. In this paper, we establish the φ -transform characterizations of the spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$; via these characterizations, we also obtain their embedding properties (which on $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is also sharp), smooth atomic and molecular decomposition characterizations for suitable τ . Moreover, using their atomic and molecular decomposition characterizations, we investigate the trace properties and the boundedness of pseudo-differential operators with homogeneous symbols (see [16]) in $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which generalizes the corresponding classical results on homogeneous Besov and Triebel-Lizorkin spaces when $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$; see, for example, Jawerth [17, Theorem 5.1] and [18, Theorem 2.1] (or Frazier-Jawerth [12, Theorem 11.1]), and Grafakos-Torres [16, Theorems 1.1 and 1.2]. Recall that the study of pseudo-differential operators with non-homogeneous symbols on non-homogeneous Besov and Triebel-Lizorkin spaces using φ -transform arguments was started by Torres [23, 24]; the results in [16] are based on these works. See also those articles for other references to previous work on pseudo-differential operators on Triebel-Lizorkin spaces using more classical methods. We will concentrate here on φ -transform arguments.

To recall the definitions of $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [30, 31], we need some notation. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of all Schwartz functions on \mathbb{R}^n . Following Triebel's [25], set

$$\mathcal{S}_\infty(\mathbb{R}^n) \equiv \left\{ \varphi \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \varphi(x) x^\gamma dx = 0 \text{ for all multi-indices } \gamma \in (\mathbb{N} \cup \{0\})^n \right\}$$

and use $\mathcal{S}'_\infty(\mathbb{R}^n)$ to denote the topological dual of $\mathcal{S}_\infty(\mathbb{R}^n)$, namely, the set of all continuous linear functionals on $\mathcal{S}_\infty(\mathbb{R}^n)$ endowed with weak $*$ -topology. Recall that $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}(\mathbb{R}^n)$ and $\mathcal{S}'_\infty(\mathbb{R}^n)$ are topologically equivalent, where $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ denote, respectively, the space of all Schwartz distributions and the set of all polynomials on \mathbb{R}^n .

For each cube Q in \mathbb{R}^n , we denote its side length by $\ell(Q)$, its center by c_Q , and set $jQ \equiv -\log_2 \ell(Q)$. For $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $j \in \mathbb{Z}$, let Q_{jk} be the dyadic cube $\{(x_1, \dots, x_n) : k_i \leq 2^j x_i < k_i + 1 \text{ for } i = 1, \dots, n\} \subset \mathbb{R}^n$, x_Q be the *lower left-corner* $2^{-j}k$ of $Q = Q_{jk}$, $\mathcal{D}(\mathbb{R}^n) \equiv \{Q_{jk}\}_{j,k}$ and $\mathcal{D}_j(\mathbb{R}^n) \equiv \{Q \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) = 2^{-j}\}$. When dyadic cube Q appears as an index, such as $\sum_{Q \in \mathcal{D}(\mathbb{R}^n)}$ and $\{\cdot\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, it is understood that Q runs over all *dyadic cubes* in \mathbb{R}^n .

For $x \in \mathbb{R}^n$ and $r > 0$, we write $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$. Next we recall the notion of Hausdorff capacities; see, for example, [1, 29]. Let $E \subset \mathbb{R}^n$ and $d \in (0, n]$. The *d-dimensional Hausdorff capacity* of E is defined by

$$H^d(E) \equiv \inf \left\{ \sum_j r_j^d : E \subset \bigcup_j B(x_j, r_j) \right\},$$

where the infimum is taken over all covers $\{B(x_j, r_j)\}_{j=1}^\infty$ of E by countable families of open balls. It is well-known that H^d is monotone, countably subadditive and vanishes on

empty set. Moreover, the notion of H^d can be extended to $d = 0$. In this case, H^0 has the property that for all sets $E \subset \mathbb{R}^n$, $H^0(E) \geq 1$, and $H^0(E) = 1$ if and only if E is bounded.

For any function $f : \mathbb{R}^n \mapsto [0, \infty]$, the *Choquet integral of f with respect to H^d* is defined by

$$\int_{\mathbb{R}^n} f dH^d \equiv \int_0^\infty H^d(\{x \in \mathbb{R}^n : f(x) > \lambda\}) d\lambda.$$

This functional is not sublinear, so sometimes we need to use an equivalent integral with respect to the d -dimensional dyadic Hausdorff capacity \tilde{H}^d , which is sublinear; see [29] (also [30, 31]) for the definition of dyadic Hausdorff capacities and their properties.

Set $\mathbb{R}_+^{n+1} \equiv \mathbb{R}^n \times (0, \infty)$. For any measurable function ω on \mathbb{R}_+^{n+1} and $x \in \mathbb{R}^n$, we define its *nontangential maximal function* $N\omega(x)$ by setting $N\omega(x) \equiv \sup_{|y-x|<t} |\omega(y, t)|$.

In what follows, for any $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we use $\mathcal{F}\varphi$ to denote its Fourier transform, namely, for all $\xi \in \mathbb{R}^n$, $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^n} e^{-i\xi x} \varphi(x) dx$. For all $j \in \mathbb{Z}$ and $x \in \mathbb{R}^n$, let $\varphi_j(x) \equiv 2^{jn} \varphi(2^j x)$. For any $p, q \in (0, \infty]$, let $(p \vee q) \equiv \max\{p, q\}$; and for any $t \in [1, \infty]$, we denote by t' the conjugate index, namely, $1/t + 1/t' = 1$.

We now recall the notions of $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [30, Definition 5.2] and [31, Definition 6.1].

Definition 1.1. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \mathcal{F}\varphi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$ and $\mathcal{F}\varphi$ never vanishes on $\{\xi \in \mathbb{R}^n : 3/5 \leq |\xi| \leq 5/3\}$. Let $p \in (1, \infty)$ and $s \in \mathbb{R}$.

(i) If $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$, the *Besov-Hausdorff space* $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \|\varphi_j * f[\omega(\cdot, 2^{-j})]^{-1}\|_{L^p(\mathbb{R}^n)}^q \right\}^{\frac{1}{q}} < \infty,$$

where ω runs over all nonnegative Borel measurable functions on \mathbb{R}_+^{n+1} such that

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \leq 1 \quad (1.1)$$

and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes.

(ii) If $q \in (1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$, the *Triebel-Lizorkin-Hausdorff space* $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that

$$\|f\|_{F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} |\varphi_j * f[\omega(\cdot, 2^{-j})]^{-1}|^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty,$$

where ω runs over all nonnegative Borel measurable functions on \mathbb{R}_+^{n+1} such that ω satisfies (1.1) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\varphi_j * f$ vanishes.

To simplify the presentation, in what follows, we use $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. When $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then it will be understood tacitly that $q \in (1, \infty)$. It was proved in [30, Proposition 5.1] and [31, Section 6]

that the space $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ . We also remark that when $\tau = 0$, then $B\dot{H}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^s(\mathbb{R}^n)$; when $\alpha \in (0, 1)$, $s = -\alpha$, $p = q = 2$ and $\tau = 1/2 - \alpha/n$, then $A\dot{H}_{2,2}^{-\alpha,1/2-\alpha}(\mathbb{R}^n) \equiv HH_{-\alpha}^1(\mathbb{R}^n)$, which is the predual space of $Q_\alpha(\mathbb{R}^n)$.

We now recall the notions of Besov-type spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Triebel-Lizorkin-type spaces $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [31, Definition 1.1] and [30, Definition 3.2].

Definition 1.2. Let $s \in \mathbb{R}$, $\tau \in [0, \infty)$, $q \in (0, \infty]$ and φ be as in Definition 1.1.

(i) If $p \in (0, \infty]$, the Besov-type space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} \left[\int_P (2^{js} |\varphi_j * f(x)|)^p dx \right]^{q/p} \right\}^{1/q}$$

with suitable modifications made when $p = \infty$ or $q = \infty$.

(ii) If $p \in (0, \infty)$, the Triebel-Lizorkin-type space $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{j=j_P}^{\infty} (2^{js} |\varphi_j * f(x)|)^q dx \right]^{p/q} \right\}^{1/p}$$

with suitable modifications made when $q = \infty$.

Similarly, we use $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. If $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ means $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then the case $p = \infty$ is excluded. It was proved in [31, Corollary 3.1] that the space $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is independent of the choices of φ . Also, [30, Theorem 5.1] and [31, Theorem 6.1] show that $(A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = \dot{A}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$. This result partially extends the well-known dual results on Besov spaces, Triebel-Lizorkin spaces and the recent result that $(HH_{-\alpha}^1(\mathbb{R}^n))^* = Q_\alpha(\mathbb{R}^n)$ obtained in [6, Theorem 7.1].

We remark that when $\tau = 0$, then $\dot{B}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,0}(\mathbb{R}^n) \equiv \dot{F}_{p,q}^s(\mathbb{R}^n)$; when $\alpha \in (0, 1)$, $s = \alpha$, $p = q = 2$ and $\tau = 1/2 - \alpha/n$, then $\dot{A}_{2,2}^{\alpha,1/2-\alpha}(\mathbb{R}^n) \equiv Q_\alpha(\mathbb{R}^n)$; see [30, Corollary 3.1]. It was proved in [22] that Besov-Morrey spaces in [21] are proper subspaces of $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and that Triebel-Lizorkin-Morrey spaces in [21] are special cases of $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. It was also proved in [21] that Morrey spaces are special cases of Triebel-Lizorkin-Morrey spaces. The φ -transform characterizations, embedding properties, smooth atomic and molecular decomposition characterizations of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were obtained in [31], which were further applied in [22] to establish their trace properties and the boundedness of pseudo-differential operators with homogeneous symbols in $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

In Section 2 of this paper, we establish the φ -transform characterizations (see Theorem 2.1 below) and embedding properties (Proposition 2.2 below) of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. In particular, we show, in Proposition 2.3 below, that the embedding property of $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is sharp. Using these φ -transform characterizations, in Section 3 below, we obtain the boundedness of almost diagonal operators and the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. As applications of these decomposition characterizations, in

Section 4 of this paper, we investigate the trace properties (see Theorem 4.2 below) and the boundedness of pseudo-differential operators with homogeneous symbols in $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ (see Theorem 4.1 below). We pointed out that the method used in the proof of Theorem 4.1 comes from [14, 9, 23, 24, 16].

Notice that the spaces $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are only known to be quasi-normed spaces so far due to the infimum on ω appearing in their definitions, which satisfies the condition (1.1). This brings us some essential difficulties, comparing with the methods used in [31, 22] for the spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$. To overcome these new difficulties, we use the Aoki theorem (see [2] and the proof of Theorem 3.1 below) and establish some subtly equivalent characterizations on the Hausdorff capacity (see Lemmas 2.4, 3.1 and 4.1 below). These characterizations on the Hausdorff capacity are geometrical, whose proofs are constructive and invoke some covering lemmas. Propositions 2.2 and 2.3 and Theorem 3.1 below reflect the differences between the spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $F\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and the spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$; see also Remarks 2.3 and 3.1 below.

Finally, we make some conventions on notation. Throughout the whole paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. The symbol $A \lesssim B$ means that $A \leq CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. If E is a subset of \mathbb{R}^n , we denote by χ_E the characteristic function of E . For all $Q \in \mathcal{D}(\mathbb{R}^n)$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, set $\varphi_Q(x) \equiv |Q|^{-1/2}\varphi(2^jQ(x - x_Q))$ and $\tilde{\chi}_Q(x) \equiv |Q|^{-1/2}\chi_Q(x)$ for all $x \in \mathbb{R}^n$. We also set $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv (\mathbb{N} \cup \{0\})$.

2 The φ -transform characterizations

In this section, we establish the φ -transform characterizations of the spaces $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in the sense of Frazier and Jawerth; see, for example, [10, 11, 12, 13]. We begin with the definition of the corresponding sequence space of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition 2.1. Let $p \in (1, \infty)$ and $s \in \mathbb{R}$.

(i) If $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$, the sequence space $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that

$$\|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q| \tilde{\chi}_Q[\omega(\cdot, 2^{-j})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right\}^{\frac{1}{q}} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions ω on \mathbb{R}_+^{n+1} such that ω satisfies (1.1) and with the restriction that for any $j \in \mathbb{Z}$, $\omega(\cdot, 2^{-j})$ is allowed to vanish only where $\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q| \tilde{\chi}_Q$ vanishes.

(ii) If $q \in (1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)'}]$, the sequence space $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is then defined to be the set of all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that

$$\|t\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left(\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q| \tilde{\chi}_Q[\omega(\cdot, 2^{-j})]^{-1} \right)^q \right\} \right\|_{L^p(\mathbb{R}^n)}^{\frac{1}{q}} < \infty,$$

where the infimum is taken over all nonnegative Borel measurable functions ω on \mathbb{R}_+^{n+1} with the same restrictions as in (i).

Similarly, in what follows, we use $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to denote either $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ or $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. When $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ denotes $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then it will be understood tacitly that $q \in (1, \infty)$. We remark that $\|\cdot\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is a quasi-norm, namely, there exists a nonnegative constant $\rho \in [0, 1]$ such that for all $t_1, t_2 \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,

$$\|t_1 + t_2\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq 2^\rho (\|t_1\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} + \|t_2\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}). \quad (2.1)$$

Remark 2.1. On (1.1), we observe that if $0 < a \leq b \leq \frac{1}{\tau}$, then for all nonnegative measurable functions ω on \mathbb{R}_+^{n+1} , $\int_{\mathbb{R}^n} [N\omega(x)]^a dH^{n\tau a}(x) < \infty$ induces $\int_{\mathbb{R}^n} [N\omega(x)]^b dH^{n\tau b}(x) < \infty$. In fact, without loss of generality, we may assume that $\int_{\mathbb{R}^n} [N\omega(x)]^a dH^{n\tau a}(x) \leq 1$. For all $l \in \mathbb{Z}$, set $E_l \equiv \{x \in \mathbb{R}^n : N\omega(x) > 2^l\}$. Then

$$1 \geq \int_{\mathbb{R}^n} [N\omega(x)]^a dH^{n\tau a}(x) \sim \sum_{l \in \mathbb{Z}} 2^{la} H^{n\tau a}(E_l).$$

For each $l \in \mathbb{Z}$, we choose a ball covering $\{B(x_{jl}, r_{jl})\}_j$ of E_l that almost attains $H^{n\tau a}(E_l)$: $H^{n\tau a}(E_l) \sim \sum_j r_{jl}^{n\tau a}$. Thus, $\sum_{l \in \mathbb{Z}} 2^{la} \sum_j r_{jl}^{n\tau a} \lesssim 1$, and hence, for all j and l , $2^l r_{jl}^{n\tau} \lesssim 1$. Then $2^{lb} r_{il}^{n\tau b} \lesssim 2^{la} r_{il}^{n\tau a}$ since $a \leq b$ and

$$\int_{\mathbb{R}^n} [N\omega(x)]^b dH^{n\tau b}(x) \sim \sum_{l \in \mathbb{Z}} 2^{lb} H^{n\tau b}(E_l) \lesssim \sum_{l \in \mathbb{Z}} 2^{lb} \sum_j r_{jl}^{n\tau b} \lesssim \sum_{l \in \mathbb{Z}} 2^{la} \sum_j r_{jl}^{n\tau a},$$

which yields the above claim.

Let φ be as in Definition 1.1. For all $x \in \mathbb{R}^n$, set $\tilde{\varphi}(x) \equiv \overline{\varphi(x)}$. Then by [13, Lemma (6.9)], there exists a function $\psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, $\mathcal{F}\psi$ never vanishes on $\{\xi \in \mathbb{R}^n : 3/5 \leq |\xi| \leq 5/3\}$ and that for all $\xi \in \mathbb{R}^n$, $\sum_{j \in \mathbb{Z}} \mathcal{F}\tilde{\varphi}(2^{-j}\xi) \mathcal{F}\psi(2^{-j}\xi) = \chi_{\mathbb{R}^n \setminus \{0\}}(\xi)$. Furthermore, we have the following *Calderón reproducing formula* which asserts that for all $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$,

$$f = \sum_{j \in \mathbb{Z}} \psi_j * \tilde{\varphi}_j * f = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_Q \rangle \psi_Q \quad (2.2)$$

in $\mathcal{S}'_\infty(\mathbb{R}^n)$; see [31, Lemma 2.1].

Now we recall the notion of the φ -transform; see, for example, [10, 11, 12, 13].

Definition 2.2. Let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp } \mathcal{F}\varphi, \text{supp } \mathcal{F}\psi \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$, $\mathcal{F}\varphi, \mathcal{F}\psi$ never vanish on $\{\xi \in \mathbb{R}^n : 3/5 \leq |\xi| \leq 5/3\}$ and $\sum_{j \in \mathbb{Z}} \mathcal{F}(\tilde{\varphi}_j) \mathcal{F}(\psi_j) \equiv \chi_{\mathbb{R}^n \setminus \{0\}}$.

(i) The φ -transform S_φ is defined to be the map taking each $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ to the sequence $S_\varphi f \equiv \{(S_\varphi f)_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, where $(S_\varphi f)_Q \equiv \langle f, \varphi_Q \rangle$ for all $Q \in \mathcal{D}(\mathbb{R}^n)$.

(ii) The *inverse* φ -transform T_ψ is defined to be the map taking a sequence $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ to $T_\psi t \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q \psi_Q$.

To show that T_ψ is well defined for all $t \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we need the following conclusion.

Lemma 2.1. Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, \frac{1}{(p \vee q)\gamma}]$. Then for all $t \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $T_\psi t = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q \psi_Q$ converges in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, $T_\psi : a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \rightarrow \mathcal{S}'_\infty(\mathbb{R}^n)$ is continuous.

Proof. By similarity, we only consider the space $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We need to show that there exists an $M \in \mathbb{Z}_+$ such that for all $\phi \in \mathcal{S}_\infty(\mathbb{R}^n)$, $\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |t_Q| |\langle \psi_Q, \phi \rangle| \lesssim \|\phi\|_{\mathcal{S}_M}$, where and in what follows, for all $M \in \mathbb{Z}_+$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we set $\|\varphi\|_{\mathcal{S}_M} \equiv \sup_{|\gamma| \leq M} \sup_{x \in \mathbb{R}^n} |\partial^\gamma \varphi(x)| (1 + |x|)^{n+M+|\gamma|}$.

Choose a Borel function ω that almost attains the infimum in Definition 2.1 (i). That is, ω is a function on \mathbb{R}_+^{n+1} satisfying (1.1) as well as

$$\left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left\| \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q| \tilde{\chi}_Q[\omega(\cdot, 2^{-j})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right\}^{\frac{1}{q}} \leq 2 \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \quad (2.3)$$

A simple consequence obtained from (1.1) is that for all $(x, s) \in \mathbb{R}_+^{n+1}$, $\omega(x, s) \lesssim s^{-n\tau}$; see [30, Remark 4.1]. Then for all $Q \in \mathcal{D}_j(\mathbb{R}^n)$, by Hölder's inequality and (2.3), we have

$$|t_Q| \leq |Q|^{-\tau - \frac{1}{p}} |t_Q| \left(\int_Q [\omega(x, 2^{-j})]^{-p} dx \right)^{\frac{1}{p}} \lesssim |Q|^{\frac{s}{n} + \frac{1}{2} - \tau - \frac{1}{p}} \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \quad (2.4)$$

Recall that as a special case of [4, Lemma 2.11], there exists a positive constant L_0 such that for all $j \in \mathbb{Z}$,

$$\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \left(1 + \frac{|x_Q|^n}{\max\{1, |Q|\}} \right)^{-L_0} \lesssim 2^{n|j|}. \quad (2.5)$$

Furthermore, it was proved in [31, p. 10] that if $L > \max\{1/p + 1/2 - s/n - \tau, 1/p + 3/2 + s/n + \tau, L_0\}$, then there exists an $M \in \mathbb{Z}_+$ such that for all $Q \in \mathcal{D}_j(\mathbb{R}^n)$,

$$|\langle \psi_Q, \phi \rangle| \lesssim \|\phi\|_{\mathcal{S}_M} \left(1 + \frac{|x_Q|^n}{\max\{1, |Q|\}} \right)^{-L} (\min\{2^{-jn}, 2^{jn}\})^L; \quad (2.6)$$

see also [4, (3.18)]. Using (2.4), (2.6) and (2.5), we conclude that

$$\begin{aligned} \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |t_Q| |\langle \psi_Q, \phi \rangle| &\lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \|\phi\|_{\mathcal{S}_M} \\ &\quad \times \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} |Q|^{\frac{s}{n} + \frac{1}{2} - \tau - \frac{1}{p}} \left(1 + \frac{|x_Q|^n}{\max\{1, |Q|\}} \right)^{-L} 2^{-L|j_Q|n} \\ &\lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \|\phi\|_{\mathcal{S}_M}, \end{aligned}$$

which completes the proof of Lemma 2.1. \square

Now we are ready to present our main result of this section.

Theorem 2.1. *Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $\tau \in [0, \frac{1}{(p \vee q)\tau}]$, φ and ψ be as in Definition 2.2. Then $S_\varphi : A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \rightarrow a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $T_\psi : a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \rightarrow A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded; moreover, $T_\psi \circ S_\varphi$ is the identity on $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.*

To prove Theorem 2.1, we need some technical lemmas. For a sequence $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, $Q \in \mathcal{D}(\mathbb{R}^n)$, $r \in (0, \infty]$ and $\lambda \in (0, \infty)$, define

$$(t_{r,\lambda}^*)_Q \equiv \left(\sum_{P \in \mathcal{D}_{j_Q}(\mathbb{R}^n)} \frac{|t_P|^r}{(1 + [\ell(P)]^{-1}|x_P - x_Q|)^\lambda} \right)^{\frac{1}{r}}$$

and $t_{r,\lambda}^* \equiv \{(t_{r,\lambda}^*)_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$. For any $p, q \in (0, \infty]$, let $p \wedge q \equiv \min\{p, q\}$. The following estimate is crucial in that this corresponds to the maximal operator estimate.

Lemma 2.2. *Let s, p, q, τ be as in Theorem 2.1 and $\lambda \in (n, \infty)$ be sufficiently large. Then there exists a positive constant C such that for all $t \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|t_{p \wedge q, \lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.*

Proof. The inequality $\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|t_{p \wedge q, \lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ being trivial, we only need to concentrate on $\|t_{p \wedge q, \lambda}^*\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$. Also, by similarity, we only consider the spaces $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We choose a Borel function ω as in the proof of Lemma 2.1. For all cubes $Q \in \mathcal{D}_j(\mathbb{R}^n)$ and $m \in \mathbb{N}$, we set $A_0(Q) \equiv \{P \in \mathcal{D}_j(\mathbb{R}^n) : 2^j|x_P - x_Q| \leq 1\}$ and $A_m(Q) \equiv \{P \in \mathcal{D}_j(\mathbb{R}^n) : 2^{m-1} < 2^j|x_P - x_Q| \leq 2^m\}$. The triangle inequality that $|x - y| \leq |x - x_Q| + |x_Q - x_P| + |x_P - y|$ gives us that $|x - y| \leq 3\sqrt{n}2^{m-j}$ provided $x \in Q$, $y \in P$ and $P \in A_m(Q)$.

For all $m \in \mathbb{Z}_+$ and $(x, s) \in \mathbb{R}_+^{n+1}$, we set

$$\omega_m(x, s) \equiv 2^{-mn(\lfloor (p \vee q)' \rfloor + 2)} \sup\{\omega(y, s) : y \in \mathbb{R}^n, |y - x| < \sqrt{n}2^{m+2}s\},$$

where and in what follows, $\lfloor s \rfloor$ denotes the *maximal integer no more than s* . By the argument in [30, Lemma 5.2], we know that ω_m still satisfies (1.1) modulo multiplicative constants independent of m . Also it follows from the definition of ω_m that for all $x \in Q$, $y \in P$ with $P \in A_m(Q)$, $\omega(y, 2^{-j}) \leq 2^{mn(\lfloor (p \vee q)' \rfloor + 2)}\omega_m(x, 2^{-j})$. For all $r \in (0, \infty)$ and $a \in (0, r)$, using this estimate and the monotonicity of $l^{a/r}$, we obtain that for all $x \in Q$,

$$\begin{aligned} & \sum_{P \in A_m(Q)} \frac{|t_P|^r}{(1 + 2^j|x_Q - x_P|)^\lambda} [\omega_m(x, 2^{-j})]^{-r} \\ & \leq \left\{ \sum_{P \in A_m(Q)} \frac{|t_P|^a}{(1 + 2^j|x_Q - x_P|)^{\lambda a/r}} [\omega_m(x, 2^{-j})]^{-a} \right\}^{r/a} \\ & \lesssim 2^{-m\lambda + jnr/a} \left\{ \int_{\mathbb{R}^n} \sum_{P \in A_m(Q)} |t_P|^a \chi_P(y) [\omega_m(x, 2^{-j})]^{-a} dy \right\}^{r/a} \\ & \lesssim 2^{-m\lambda + nr\{j/a + m(\lfloor (p \vee q)' \rfloor + 2)\}} \left\{ \int_{\mathbb{R}^n} \sum_{P \in A_m(Q)} |t_P|^a \chi_P(y) [\omega(y, 2^{-j})]^{-a} dy \right\}^{r/a} \\ & \lesssim 2^{-m\lambda + mnr(1/a + \lfloor (p \vee q)' \rfloor + 2)} \left\{ \text{HL} \left(\sum_{P \in A_m(Q)} |t_P|^a \chi_P[\omega(\cdot, 2^{-j})]^{-a} \right) (x) \right\}^{r/a}, \end{aligned}$$

where HL denotes the Hardy-Littlewood maximal operator on \mathbb{R}^n .

For all $m \in \mathbb{Z}_+$, set $t_{r,\lambda}^{*,m} \equiv \{(t_{r,\lambda}^{*,m})_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ with

$$(t_{r,\lambda}^{*,m})_Q \equiv \left(\sum_{P \in A_m(Q)} \frac{|t_P|^r}{(1 + [\ell(P)]^{-1}|x_P - x_Q|)^\lambda} \right)^{\frac{1}{r}}.$$

In what follows, choose $a \in (0, p \wedge q)$ and $\lambda > (p \wedge q)[n(1/a + \lfloor (p \vee q)' \rfloor + 2) + \rho]$, where ρ is a nonnegative constant as in (2.1). By (2.1), the previous pointwise estimate and the $L^{\frac{p}{a}}(\mathbb{R}^n)$ -boundedness of HL, we obtain

$$\begin{aligned} & \|t_{p \wedge q, \lambda}^*\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\ & \lesssim \sum_{m=0}^{\infty} 2^{\rho m} \|t_{p \wedge q, \lambda}^{*,m}\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\ & \lesssim \sum_{m=0}^{\infty} 2^{\rho m} \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left[\int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \left(\sum_{P \in A_m(Q)} \frac{|t_P|^{p \wedge q}}{(1 + [\ell(P)]^{-1}|x_P - x_Q|)^\lambda} \right)^{\frac{p}{p \wedge q}} \right. \right. \\ & \quad \left. \left. \times \frac{\tilde{\chi}_Q(x)^p}{[\omega_m(x, 2^{-j})]^p} dx \right]^{\frac{a}{p}} \right\}^{\frac{1}{q}} \\ & \lesssim \sum_{m=0}^{\infty} 2^{-\frac{m}{p \wedge q} \{\lambda - (p \wedge q)[n(1/a + \lfloor (p \vee q)' \rfloor + 2) + \rho\}} \\ & \quad \times \left[\sum_{j \in \mathbb{Z}} 2^{jsq} \left\{ \int_{\mathbb{R}^n} \left[\text{HL} \left(\sum_{P \in \mathcal{D}_j(\mathbb{R}^n)} \frac{(|t_P| \tilde{\chi}_P)^a}{[\omega(\cdot, 2^{-j})]^a} \right) (x) \right]^{\frac{p}{a}} dx \right\}^{\frac{a}{p}} \right]^{\frac{1}{q}} \lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}, \end{aligned}$$

which completes the proof of Lemma 2.2. \square

For any $f \in \mathcal{S}'_\infty(\mathbb{R}^n)$, $\gamma \in \mathbb{Z}_+$ and $Q \in \mathcal{D}_j(\mathbb{R}^n)$, set $\sup_Q(f) \equiv |Q|^{1/2} \sup_{y \in Q} |\tilde{\varphi}_j * f(y)|$ and

$$\inf_{Q,\gamma}(f) \equiv |Q|^{1/2} \max \left\{ \inf_{y \in \tilde{Q}} |\tilde{\varphi}_j * f(y)| : \ell(\tilde{Q}) = 2^{-\gamma} \ell(Q), \tilde{Q} \subset Q \right\}.$$

Let $\sup(f) \equiv \{\sup_Q(f)\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ and $\inf_\gamma(f) \equiv \{\inf_{Q,\gamma}(f)\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$. We have the following conclusion, whose proof is similar to [12, Lemma 2.5] and we omit the details.

Lemma 2.3. *Let s, p, q, τ be as in Theorem 2.1 and $\gamma \in \mathbb{Z}_+$ be sufficiently large. Then there exists a constant $C \in [1, \infty)$ such that for all $f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,*

$$C^{-1} \|\inf_\gamma(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq \|\sup(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \|\inf_\gamma(f)\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

With the Calderón reproducing formula (2.2), Lemmas 2.2 and 2.3, the proof of Theorem 2.1 follows the method pioneered by Frazier and Jawerth (see [12, pp. 50-51]); see also the proof of [5, Theorem 3.5]. We omit the details.

Recall that the corresponding sequence spaces $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ of $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [31, Definition 3.1] were defined as follows.

Definition 2.3. Let $s \in \mathbb{R}$, $q \in (0, \infty]$ and $\tau \in (0, \infty)$. The sequence space $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is defined to be the set of all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ such that $\|t\|_{\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)} < \infty$, where if $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty]$, then

$$\|t\|_{\dot{b}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^{\infty} 2^{jsq} \left[\int_P \left(\sum_{l(Q)=2^{-j}} |t_Q| \tilde{\chi}_Q(x) \right)^p dx \right]^{q/p} \right\}^{1/q}$$

and if $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n) \equiv \dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)$ for $p \in (0, \infty)$, then

$$\|t\|_{\dot{f}_{p,q}^{s,\tau}(\mathbb{R}^n)} \equiv \sup_{P \in \mathcal{D}(\mathbb{R}^n)} \frac{1}{|P|^\tau} \left\{ \int_P \left[\sum_{Q \subset P} \left(|Q|^{-s/n} |t_Q| \tilde{\chi}_Q(x) \right)^q \right]^{p/q} dx \right\}^{1/p}.$$

We now establish the duality between $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, which is used in Sections 3 and 4 below. In what follows, for any quasi-Banach spaces \mathcal{B}_1 and \mathcal{B}_2 , the symbol $\mathcal{B}_1 \hookrightarrow \mathcal{B}_2$ means that there exists a positive constant C such that for all $f \in \mathcal{B}_1$, then $f \in \mathcal{B}_2$ and $\|f\|_{\mathcal{B}_2} \leq C\|f\|_{\mathcal{B}_1}$.

Proposition 2.1. Let s, p, q, τ be as in Theorem 2.1. Then $(a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ in the following sense.

If $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$, then the map

$$\lambda = \{\lambda_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \mapsto \langle \lambda, t \rangle \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \bar{t}_Q$$

defines a continuous linear functional on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ with operator norm no more than a constant multiple of $\|t\|_{\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$.

Conversely, every $L \in (a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^*$ is of this form for a certain $t \in \dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\|t\|_{\dot{a}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$ is no more than a constant multiple of the operator norm of L .

Proof. We only consider the spaces $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ because the assertion for $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ can be proved similarly. Below we write $\mathbb{R}_+^{n+1} \equiv \{(x, a) \in \mathbb{R}_+^{n+1} : \log_2 a \in \mathbb{Z}\}$.

For $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in \dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$ and $\lambda = \{\lambda_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, let F and G be functions on \mathbb{R}_+^{n+1} defined by setting, for all $x \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, $F(x, 2^{-j}) \equiv \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |\lambda_Q| \tilde{\chi}_Q$ and $G(x, 2^{-j}) \equiv \sum_{P \in \mathcal{D}_j(\mathbb{R}^n)} |t_P| \tilde{\chi}_P$. Since

$$\|F\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})} \sim \|\lambda\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

and $\|G\|_{B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_+^{n+1})} \sim \|t\|_{\dot{b}_{p',q'}^{-s,\tau}(\mathbb{R}^n)}$, where $B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1})$ and $B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_+^{n+1})$ are tent spaces introduced in [31, Definition 5.2], by the duality of tent spaces obtained in [31, Theorem 5.1] that $(B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_+^{n+1}))^* = B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_+^{n+1})$, we have

$$\left| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \bar{t}_Q \right| \leq \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \sum_{P \in \mathcal{D}_j(\mathbb{R}^n)} |\lambda_Q| \tilde{\chi}_Q(x) |t_P| \tilde{\chi}_P(x) dx$$

$$\begin{aligned}
&= \sum_{j \in \mathbb{Z}} \int_{\mathbb{R}^n} F(x, 2^{-j}) G(x, 2^{-j}) dx \lesssim \|F\|_{B\dot{T}_{p,q}^{s,\tau}(\mathbb{R}_Z^{n+1})} \|G\|_{B\dot{W}_{p',q'}^{-s,\tau}(\mathbb{R}_Z^{n+1})} \\
&\sim \|\lambda\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \|t\|_{b\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)},
\end{aligned}$$

which implies that $b\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n) \hookrightarrow (b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^*$.

Conversely, since sequences with finite non-vanishing elements are dense in $b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we know that every $L \in (b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^*$ is of the form $\lambda \mapsto \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \lambda_Q \bar{t}_Q$ for a certain $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$. It remains to show that $\|t\|_{b\dot{H}_{p',q'}^{-s,\tau}(\mathbb{R}^n)} \lesssim \|L\|_{(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^*}$.

Fix $P \in \mathcal{D}(\mathbb{R}^n)$ and $a \in \mathbb{R}$. For $j \geq j_P$, let X_j be the set of all $Q \in \mathcal{D}_j(\mathbb{R}^n)$ satisfying $Q \subset P$ and let μ be a measure on X_j such that the μ -measure of the “point” Q is $|Q|/|P|^{\tau a}$. Also, let l_P^q denote the set of all $\{a_j\}_{j \geq j_P} \subset \mathbb{C}$ with $\|\{a_j\}_{j \geq j_P}\|_{l_P^q} \equiv (\sum_{j=j_P}^\infty |a_j|^q)^{1/q}$ and $l_P^q(l^p(X_j, d\mu))$ denote the set of all $\{a_{Q,j}\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P} \subset \mathbb{C}$ with

$$\|\{a_{Q,j}\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^q(l^p(X_j, d\mu))} \equiv \left(\sum_{j=j_P}^\infty \left[\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P} |a_{Q,j}|^p \frac{|Q|}{|P|^{\tau a}} \right]^{\frac{q}{p}} \right)^{1/q}.$$

It is easy to see that the dual space of $l_P^q(l^p(X_j, d\mu))$ is $l_P^{q'}(l^{p'}(X_j, d\mu))$; see [25, p.177]. Via this observation and the already proved conclusion of this proposition, we see that

$$\begin{aligned}
&\frac{1}{|P|^\tau} \left\{ \sum_{j=j_P}^\infty \left[\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P} \left(|Q|^{-\frac{s}{n}-\frac{1}{2}} |t_Q| \right)^{p'} |Q| \right]^{\frac{q'}{p'}} \right\}^{\frac{1}{q'}} \\
&= \|\{|Q|^{-\frac{s}{n}-\frac{1}{2}} |t_Q|\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^{q'}(l^{p'}(X_j, d\mu))} \\
&= \sup_{\|\{\lambda_Q\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^q(l^p(X_j, d\mu))} \leq 1} \left| \sum_{j=j_P}^\infty \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P} \lambda_Q |Q|^{-\frac{s}{n}-\frac{1}{2}} |t_Q| |Q|/|P|^{\tau p'} \right| \\
&\leq \sup_{\|\{\lambda_Q\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^q(l^p(X_j, d\mu))} \leq 1} \|t\|_{(b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^*} \\
&\quad \times \left\| \{\lambda_Q |Q|^{-\frac{s}{n}-\frac{1}{2}} |Q|/|P|^{\tau p'}\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P} \right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.
\end{aligned}$$

To finish the proof of this proposition, it suffices to show that

$$\left\| \{\lambda_Q |Q|^{-\frac{s}{n}-\frac{1}{2}} |Q|/|P|^{\tau p'}\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P} \right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim 1$$

for all sequences λ satisfying $\|\{\lambda_Q\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^q(l^p(X_j, d\mu))} \leq 1$. In fact, let $B \equiv B(c_P, \sqrt{n}\ell(P))$ and ω be as in the proof of [30, Lemma 4.1] associated with B , then ω satisfies (1.1) and for all $x \in P$ and $j \geq j_P$, $[\omega(x, 2^{-j})]^{-1} \sim [\ell(P)]^{n\tau}$. We then obtain that

$$\left\| \{\lambda_Q |Q|^{-\frac{s}{n}-\frac{1}{2}} |Q|/|P|^{\tau p'}\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P} \right\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$$

$$\begin{aligned}
&\lesssim \left\{ \sum_{j=j_P}^{\infty} 2^{jsq} \left[\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P} |Q|^{-\frac{p}{2}} \left(|\lambda_Q| |Q|^{-\frac{s}{n} - \frac{1}{2}} \frac{|Q|}{|P|^{\tau p'}} \right)^p \int_Q [\omega(x, 2^{-j})]^{-p} dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&\sim \left\{ \sum_{j=j_P}^{\infty} \left[\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P} |\lambda_Q|^p |Q| / |P|^{\tau p'} \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\
&\sim \|\{\lambda_Q\}_{Q \in \mathcal{D}_j(\mathbb{R}^n), Q \subset P, j \geq j_P}\|_{l_P^q(l^p(X_j, d\mu))} \lesssim 1,
\end{aligned}$$

which completes the proof of Proposition 2.1. \square

Remark 2.2. By Proposition 2.1 and the φ -transform characterizations of the spaces $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in Theorem 2.1 and $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ in [31, Theorem 3.1], we also obtain the duality that $(A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n))^* = \dot{A}_{p',q'}^{-s,\tau}(\mathbb{R}^n)$. This gives other proofs of these conclusions, which are different from those in [30, Section 5] and [31, Section 6].

Applying Theorem 2.1, we establish the following Sobolev-type embedding properties of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. For the corresponding results on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$, see [25, p. 129].

Proposition 2.2. *Let $1 < p_0 < p_1 < \infty$ and $-\infty < s_1 < s_0 < \infty$. Assume in addition that $s_0 - n/p_0 = s_1 - n/p_1$.*

(i) *If $q \in [1, \infty)$ and $\tau \in [0, \min\{\frac{1}{(p_0 \vee q)', \frac{1}{(p_1 \vee q)'}\}]$ such that $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$, then $B\dot{H}_{p_0,q}^{s_0,\tau}(\mathbb{R}^n) \hookrightarrow B\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)$.*

(ii) *If $q, r \in (1, \infty)$ and $\tau \in [0, \min\{\frac{1}{(p_0 \vee r)', \frac{1}{(p_1 \vee q)'}\}]$ such that $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)'$, then $F\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n) \hookrightarrow F\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)$.*

Proof. By Theorem 2.1 and similarity, it suffices to prove the corresponding conclusions on sequence spaces $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, namely, to show that $\|t\|_{f\dot{H}_{p_1,q}^{s_1,\tau}(\mathbb{R}^n)} \lesssim \|t\|_{f\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n)}$ for all $t \in f\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n)$. When $\tau = 0$, this is a classic conclusion on Triebel-Lizorkin spaces.

In the case when $\tau > 0$, we have $(p_0 \vee r)' \leq (p_1 \vee q)'$. Let $t \in f\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n)$ and ω satisfy

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p_0 \vee r)'} dH^{n\tau(p_0 \vee r)'}(x) \leq 1 \quad (2.7)$$

and

$$\left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |t_Q|^r \chi_Q(x) [\omega(x, 2^{-j})]^{-r} \right]^{p_0/r} dx \right\}^{1/p_0} \lesssim \|t\|_{f\dot{H}_{p_0,r}^{s_0,\tau}(\mathbb{R}^n)}.$$

For all $(x, t) \in \mathbb{R}_+^{n+1}$, we set $\tilde{\omega}(x, s) \equiv \sup\{\omega(y, s) : y \in \mathbb{R}^n, |y - x| < \sqrt{ns}\}$. Then by the argument in [30, Lemma 5.2], we know that a constant multiple of $\tilde{\omega}$ also satisfies (2.7). Since $(p_0 \vee r)' \leq (p_1 \vee q)'$, Remark 2.1(i) tells us that $\tilde{\omega}$ satisfies

$$\int_{\mathbb{R}^n} [N\omega(x)]^{(p_1 \vee q)'} dH^{n\tau(p_1 \vee q)'}(x) \lesssim 1.$$

For all Q with $\ell(Q) = 2^{-j}$, set $\tilde{t}_Q \equiv |t_Q| \sup_{y \in Q} \{[\tilde{\omega}(y, 2^{-j})]^{-1}\}$. Observe that for all $x \in Q$ with $\ell(Q) = 2^{-j}$, $[\tilde{\omega}(x, 2^{-j})]^{-1} \lesssim \inf_{y \in Q} [\omega(y, 2^{-j})]^{-1}$, and hence, $\sup_{x \in Q} [\tilde{\omega}(x, 2^{-j})]^{-1} \lesssim$

$\inf_{y \in Q} [\omega(y, 2^{-j})]^{-1}$. This observation together with $p_0 < p_1$, $s_0 - n/p_0 = s_1 - n/p_1$ and the corresponding embedding property for Triebel-Lizorkin spaces (see, for example, [25, Theorem 2.7.1]) yields that

$$\begin{aligned}
& \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |t_Q|^q \chi_Q(x) [\tilde{\omega}(x, 2^{-j})]^{-q} \right]^{p_1/q} dx \right\}^{1/p_1} \\
& \leq \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |t_Q|^q \chi_Q(x) \sup_{y \in Q} \{ [\tilde{\omega}(y, 2^{-j})]^{-q} \} \right]^{p_1/q} dx \right\}^{1/p_1} \\
& = \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_1 q}{n} - \frac{q}{2}} |\tilde{t}_Q|^q \chi_Q(x) \right]^{p_1/q} dx \right\}^{1/p_1} = \|\tilde{t}\|_{\dot{f}_{p_1, q}^{s_1}(\mathbb{R}^n)}^{p_1} \lesssim \|\tilde{t}\|_{\dot{f}_{p_0, r}^{s_0}(\mathbb{R}^n)}^{p_1} \\
& \sim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |\tilde{t}_Q|^r \chi_Q(x) \right]^{p_0/r} dx \right\}^{1/p_0} \\
& \sim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |t_Q|^r \chi_Q(x) \sup_{y \in Q} \{ [\tilde{\omega}(y, 2^{-j})]^{-r} \} \right]^{p_0/r} dx \right\}^{1/p_0} \\
& \lesssim \left\{ \int_{\mathbb{R}^n} \left[\sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{s_0 r}{n} - \frac{r}{2}} |t_Q|^r \chi_Q(x) [\omega(x, 2^{-j})]^{-r} \right]^{p_0/r} dx \right\}^{1/p_0} \lesssim \|t\|_{f\dot{H}_{p_0, r}^{s_0, \tau}(\mathbb{R}^n)};
\end{aligned}$$

see [12, p. 38] for the definition of the sequence spaces $\dot{f}_{p, q}^s(\mathbb{R}^n)$. Therefore, $\|t\|_{f\dot{H}_{p_1, q}^{s_1, \tau}(\mathbb{R}^n)} \lesssim \|t\|_{f\dot{H}_{p_0, r}^{s_0, \tau}(\mathbb{R}^n)}$, which completes the proof of Proposition 2.2. \square

When $\tau = 0$, Proposition 2.2 recovers the corresponding results on $\dot{B}_{p, q}^s(\mathbb{R}^n)$ and $\dot{F}_{p, q}^s(\mathbb{R}^n)$ in [25, p. 129], which are known to be sharp; see [26, p. 207]. At the end of this section, we further show that the restriction that $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$ in Proposition 2.2(i) is also sharp. To see this, we need the following geometrical observation on the Hausdorff capacity.

Lemma 2.4. *Let $d \in (0, n]$. Suppose that $\{E_j\}_{j=1}^\infty$ are given subsets of \mathbb{R}^n such that $E_j \subset B((A_j, 0, \dots, 0), n)$, where $\{A_j\}_{j=1}^\infty$ is an increasing sequence of natural numbers satisfying that $A_1 \geq 10$ and for all $j, l \in \mathbb{N}$, $A_{j+l} - A_j \geq 4nl^{1/d}$. Then $H^d(\cup_{j=1}^\infty E_j)$ and $\sum_{j=1}^\infty H^d(E_j)$ are equivalent.*

Proof. The inequality $H^d(\cup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty H^d(E_j)$ is trivial. Let us prove the reverse inequality. To this end, let us first notice the following geometric observation that when a ball $B \equiv (x_B, r_B)$ intersects E_j and E_{j+l} for some $j, l \in \mathbb{N}$, then $2B$ engulfs $E_j, E_{j+1}, \dots, E_{j+l}$. Thus, $4r_B$ is greater than $A_{j+l} - A_j$ and hence, $r_B^d \geq ((A_{j+l} - A_j)/4)^d \geq ln^d$. Therefore, instead of using B we can use $B((A_j, 0, \dots, 0), n), \dots, B((A_{j+l}, 0, \dots, 0), n)$ to cover

E_j and E_{j+l} . Notice that $\{B((A_j, 0, \dots, 0), n)\}_{j=1}^\infty$ are disjoint. Based on these observations, without loss of generality, we may assume, in estimating $H^d(\cup_{j=1}^\infty E_j)$, that each ball in the ball covering meets only one E_j . From this, it is easy to follow that $H^d(\cup_{j=1}^\infty E_j) \gtrsim \sum_{j=1}^\infty H^d(E_j)$, which completes the proof of Lemma 2.4. \square

Lemma 2.5. *Let $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$, $\tau \in (0, \frac{1}{(p \vee q)'}]$ and $\{A_k\}_{k=1}^\infty$ be as in Lemma 2.4 such that $Q_k \equiv (A_k, 0, \dots, 0) + 2^{-k}[0, 1)^n \in \mathcal{D}_k(\mathbb{R}^n)$ for all $k \in \mathbb{N}$ (The existence of $\{A_k\}_{k=1}^\infty$ is obvious). Define $t_j \equiv \{(t_j)_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ so that $(t_j)_Q \equiv 2^{-\frac{kn}{2} - k(s - \frac{n}{p})}$ if $Q = Q_k$ and $k \in \{1, \dots, j\}$, $(t_j)_Q \equiv 0$ otherwise. Then for all $j \in \mathbb{N}$, $\|t_j\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $j^{\frac{1}{q} + \frac{1}{(p \vee q)'}}$ and $\|t_j\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$ is equivalent to $j^{\frac{1}{p} + \frac{1}{(p \vee q)'}}$.*

Proof. For the Besov-Hausdorff space, let us minimize

$$\left(\sum_{k=1}^j 2^{ksq} \left\| |(t_j)_{Q_k}| \tilde{\chi}_{Q_k} [\omega(\cdot, 2^{-k})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}}$$

under the condition (1.1). By the definition of t_j and the assumption on ω in Definition 2.1, we may assume that $\omega \equiv 0$ outside $\cup_{k=1}^j (Q_{0,(A_k,0,\dots,0)} \times \{2^{-k}\})$ and for all $Q \in \mathcal{D}_k(\mathbb{R}^n)$, $Q \subset Q_{0,(A_k,0,\dots,0)}$ and $k \in \{1, \dots, j\}$, $\sup_{x \in Q} \omega(x, 2^{-k}) = \sup_{x \in Q_k} \omega(x, 2^{-k})$, where $Q_{0,(A_j,0,\dots,0)} \equiv (A_j, 0, \dots, 0) + [0, 1)^n \in \mathcal{D}_0(\mathbb{R}^n)$. Also, by an observation similar to [31, Lemma 6.2], we can replace ω with the maximal function $\tilde{\omega}$ given by $\tilde{\omega}(x, 2^{-k}) \equiv \sup_{y \in Q_{k,x}} \omega(y, 2^{-k})$, where $k \in \{1, \dots, j\}$ and $Q_{k,x} \in \mathcal{D}_k(\mathbb{R}^n)$ is a unique cube containing x . This construction implies that $\tilde{\omega}$ equals a constant on $Q_{0,(A_k,0,\dots,0)}$ for each $k \in \{1, \dots, j\}$, namely, $\tilde{\omega}(\cdot, 2^{-k}) \equiv \alpha_k \chi_{Q_{0,(A_k,0,\dots,0)}}$. Notice that if $N\tilde{\omega}(x) \neq 0$, then $x \in B((A_k, 0, \dots, 0), n)$ for some $k \in \{1, \dots, j\}$. This combined with Lemma 2.4 yields that

$$\begin{aligned} & \int_{\mathbb{R}^n} [N\tilde{\omega}(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \\ &= \int_0^\infty H^{n\tau(p \vee q)'} \left(\left\{ x \in \left(\bigcup_{k=1}^j B((A_k, 0, \dots, 0), n) \right) : [N\tilde{\omega}(x)]^{(p \vee q)'} > \lambda \right\} \right) d\lambda \\ &\sim \sum_{k=1}^j \int_0^\infty H^{n\tau(p \vee q)'} \left(\left\{ x \in B((A_k, 0, \dots, 0), n) : [N\tilde{\omega}(x)]^{(p \vee q)'} > \lambda \right\} \right) d\lambda \\ &\sim \sum_{k=1}^j \int_{B((A_k, 0, \dots, 0), n)} [N\tilde{\omega}(x)]^{(p \vee q)'} dH^{n\tau(p \vee q)'}(x) \sim \sum_{k=1}^j (\alpha_k)^{(p \vee q)'}. \end{aligned}$$

On the other hand,

$$\left(\sum_{k=1}^j 2^{ksq} \left\| |(t_j)_{Q_k}| \tilde{\chi}_{Q_k} [\tilde{\omega}(\cdot, 2^{-k})]^{-1} \right\|_{L^p(\mathbb{R}^n)}^q \right)^{\frac{1}{q}} = \left[\sum_{k=1}^j (\alpha_k)^{-q} \right]^{\frac{1}{q}}.$$

In summary (modulo a multiplicative constant), we need to minimize $(\sum_{k=1}^j (\alpha_k)^{-q})^{\frac{1}{q}}$ under the condition $\sum_{k=1}^j (\alpha_k)^{(p \vee q)'} \lesssim 1$. This can be achieved as follows: By using the

geometric mean, we have

$$\begin{aligned} \left(\sum_{k=1}^j (\alpha_k)^{-q} \right)^{\frac{1}{q}} &\gtrsim \left(\sum_{k=1}^j (\alpha_k)^{-q} \right)^{\frac{1}{q}} \left(\sum_{k=1}^j (\alpha_k)^{(p \vee q)'} \right)^{\frac{1}{(p \vee q)'}} \\ &\gtrsim \left(j \sqrt[j]{\prod_{k=1}^j (\alpha_k)^{-q}} \right)^{\frac{1}{q}} \left(j \sqrt[j]{\prod_{k=1}^j (\alpha_k)^{(p \vee q)'}} \right)^{\frac{1}{(p \vee q)'}} \sim j^{\frac{1}{q} + \frac{1}{(p \vee q)'}}. \end{aligned}$$

In particular, $[\sum_{k=1}^j (\alpha_k)^{-q}]^{\frac{1}{q}} \sim j^{\frac{1}{q} + \frac{1}{(p \vee q)'}}$ when $\sum_{k=1}^j (\alpha_k)^{(p \vee q)'} \sim 1$ and the α_k 's are identical. Thus, for all $j \in \mathbb{N}$, $\|t_j\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim j^{\frac{1}{q} + \frac{1}{(p \vee q)'}}$.

For Triebel-Lizorkin-Hausdorff space, similarly to the above arguments, we see that

$$\begin{aligned} &\left(\int_{\mathbb{R}^n} \left[\sum_{k=1}^j |Q_k|^{-(s/n+1/2)q} |(t_j)_{Q_k}|^q \chi_{Q_k}(x) [\tilde{\omega}(x, 2^{-k})]^{-q} \right]^{p/q} dx \right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^n} \sum_{k=1}^j |Q_k|^{-(s/n+1/2)p} |(t_j)_{Q_k}|^p \chi_{Q_k}(x) (\alpha_k)^{-p} dx \right)^{\frac{1}{p}} = \left[\sum_{k=1}^j (\alpha_k)^{-p} \right]^{1/p}. \end{aligned}$$

Applying the geometric mean again, we have

$$\begin{aligned} \left(\sum_{k=1}^j (\alpha_k)^{-p} \right)^{\frac{1}{p}} &\gtrsim \left(\sum_{k=1}^j (\alpha_k)^{-p} \right)^{\frac{1}{p}} \left(\sum_{k=1}^j (\alpha_k)^{(p \vee q)'} \right)^{\frac{1}{(p \vee q)'}} \\ &\gtrsim \left(j \sqrt[j]{\prod_{k=1}^j (\alpha_k)^{-p}} \right)^{\frac{1}{p}} \left(j \sqrt[j]{\prod_{k=1}^j (\alpha_k)^{(p \vee q)'}} \right)^{\frac{1}{(p \vee q)'}} \sim j^{\frac{1}{p} + \frac{1}{(p \vee q)'}}. \end{aligned}$$

In particular, $[\sum_{k=1}^j (\alpha_k)^{-p}]^{\frac{1}{p}} \sim j^{\frac{1}{p} + \frac{1}{(p \vee q)'}}$ when $\sum_{k=1}^j (\alpha_k)^{(p \vee q)'} \sim 1$ and the α_k 's are identical, which implies that for all $j \in \mathbb{N}$, $\|t_j\|_{f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \sim j^{\frac{1}{p} + \frac{1}{(p \vee q)'}}$. This finishes the proof of Lemma 2.5. \square

Proposition 2.3. *Let s, τ, p_0, p_1, q, r be as in Proposition 2.2.*

- (i) *If $b\dot{H}_{p_0,q}^{s_0,\tau} \hookrightarrow b\dot{H}_{p_1,q}^{s_1,\tau}$, then $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$.*
- (ii) *If $f\dot{H}_{p_0,r}^{s_0,\tau} \hookrightarrow f\dot{H}_{p_1,q}^{s_1,\tau}$, then $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)' + \tau(\frac{1}{p_0} - \frac{1}{p_1})(p_0 \vee r)'(p_1 \vee q)'$.*

Proof. By similarity, we only consider the Besov-Hausdorff space. Let t_j be as in Lemma 2.5 with s, p replaced, respectively, by s_0 and p_0 . Since $s_0 - n/p_0 = s_1 - n/p_1$, by Lemma 2.5, we have $\|t_j\|_{b\dot{H}_{p_0,q}^{s_0,\tau}} \sim j^{\frac{1}{q} + \frac{1}{(p_0 \vee q)'}}$ and $\|t_j\|_{b\dot{H}_{p_1,q}^{s_1,\tau}} \sim j^{\frac{1}{q} + \frac{1}{(p_1 \vee q)'}}$ for all $j \in \mathbb{N}$, which together with $b\dot{H}_{p_0,q}^{s_0,\tau} \hookrightarrow b\dot{H}_{p_1,q}^{s_1,\tau}$ implies that $j^{\frac{1}{q} + \frac{1}{(p_1 \vee q)'}} \lesssim j^{\frac{1}{q} + \frac{1}{(p_0 \vee q)'}}$ for all $j \in \mathbb{N}$. Therefore, $(p_0 \vee q)' \leq (p_1 \vee q)'$. Meanwhile it is trivial that $(p_0 \vee q)' \geq (p_1 \vee q)'$ since $p_1 > p_0$. We then have $(p_0 \vee q)' = (p_1 \vee q)'$. This finishes the proof of Proposition 2.3. \square

Remark 2.3. Comparing Proposition 2.2 herein with [31, Proposition 3.3] on the space $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we see that the restriction $\tau(p_0 \vee q)' = \tau(p_1 \vee q)'$ in Proposition 2.2(i) is additional. To be surprising, Proposition 2.3(i) implies that this restriction is also necessary, and sharp in this sense. However, it is still unclear if the restriction $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)'$ in Proposition 2.2(ii) can be replaced by the restriction $\tau(p_0 \vee r)' \leq \tau(p_1 \vee q)' + \tau(\frac{1}{p_0} - \frac{1}{p_1})(p_0 \vee r)'(p_1 \vee q)'$.

3 Smooth atomic and molecular decompositions

We begin with considering the boundedness of almost diagonal operators on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which is applied to establish the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We remark that the corresponding results in $\dot{a}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{A}_{p,q}^{s,\tau}(\mathbb{R}^n)$ were already obtained in [31, Section 4].

Definition 3.1. Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $\tau \in [0, \frac{1}{(p \vee q)r}]$ and $\varepsilon \in (0, \infty)$. For all $Q, P \in \mathcal{D}(\mathbb{R}^n)$, define

$$\omega_{QP}(\varepsilon) \equiv \left(\frac{\ell(Q)}{\ell(P)} \right)^s \left(1 + \frac{|x_P - x_Q|}{\max(\ell(Q), \ell(P))} \right)^{-n-\varepsilon} \min \left(\left(\frac{\ell(P)}{\ell(Q)} \right)^{\frac{n+\varepsilon}{2}}, \left(\frac{\ell(Q)}{\ell(P)} \right)^{\frac{n+\varepsilon}{2}} \right).$$

An operator A associated with a matrix $\{a_{QP}\}_{Q,P \in \mathcal{D}(\mathbb{R}^n)}$, namely, for all sequences $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$, $At \equiv \{(At)_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \equiv \{\sum_{P \in \mathcal{D}(\mathbb{R}^n)} a_{QP} t_P\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$, is called ε -almost diagonal on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if the matrix $\{a_{QP}\}_{Q,P \in \mathcal{D}(\mathbb{R}^n)}$ satisfies

$$\sup_{Q, P \in \mathcal{D}(\mathbb{R}^n)} |a_{QP}| / \omega_{QP}(\varepsilon) < \infty.$$

We remark that any ε -almost diagonal operator on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is also an almost diagonal operator introduced by Frazier and Jawerth in [12] with $J \equiv n$. Moreover, Frazier and Jawerth proved that all almost diagonal operators are bounded on $\dot{b}_{p,q}^s(\mathbb{R}^n)$ and $\dot{f}_{p,q}^s(\mathbb{R}^n)$, which are the corresponding sequence spaces of $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$; see [11, 12, 13]. These results when $p \in (1, \infty)$ and $q \in [1, \infty)$ are generalized into the following conclusions.

Theorem 3.1. Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $\varepsilon \in (0, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)r}]$. Then all the ε -almost diagonal operators on $a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ are bounded if $\varepsilon > 2n\tau$.

To prove this theorem, we need some technical lemmas.

Lemma 3.1. Let $d \in (0, n]$ and Ω be an open set in \mathbb{R}^n such that $\Omega = \cup_{j=1}^{\infty} B_j$, where $\{B_j\}_{j=1}^{\infty} \equiv \{B(X_j, R_j)\}_{j=1}^{\infty}$ is a countable collection of balls. Define

$$\begin{aligned} & H^d(\Omega, \{B_j\}_{j=1}^{\infty}) \\ & \equiv \inf \left\{ \sum_{k=1}^{\infty} r_k^d : \Omega \subset \bigcup_{k=1}^{\infty} B(x_k, r_k), B(x_k, r_k) \supset B_j \text{ if } B_j \cap B(x_k, r_k) \neq \emptyset \right\}. \end{aligned}$$

Then there exists a positive constant C , independent of Ω , $\{B_j\}_{j=1}^{\infty}$ and d , such that

$$H^d(\Omega) \leq H^d(\Omega, \{B_j\}_{j=1}^{\infty}) \leq C(46)^d H^d(\Omega).$$

Proof. The first inequality is trivial. We only need to prove the second one. Without loss of generality, we may assume $\sup_{j \in \mathbb{N}} R_j < \infty$. By the well-known $(5r)$ -covering lemma (see, for example, [7, Theorem 2.19]), there exists a subset J^* of \mathbb{N} such that $\bigcup_{j=1}^{\infty} (3B_j) \subset \bigcup_{j \in J^*} (15B_j)$ and $\chi_{j \in J^*} \chi_{(3B_j)} \leq 1$. Furthermore, by its construction, if $B_{j'}$, $j' \in \mathbb{N}$, intersects B_j for some $j \in J^*$, we have that $(3B_{j'}) \subset (15B_j)$.

Let $\{B(x_k, r_k)\}_{k \in \mathbb{N}}$ be a collection of balls such that $\Omega \subset \bigcup_{k=1}^{\infty} B(x_k, r_k)$ and $\sum_{k=1}^{\infty} r_k^d \leq 2H^d(\Omega)$. Set

$$K_1 \equiv \{k \in \mathbb{N} : \text{When } B(x_k, 45r_k) \cap B_j \neq \emptyset \text{ for any } j \in \mathbb{N}, \text{ then } r_k \geq 135R_j\}$$

and $J_1 \equiv \{j \in \mathbb{N} : B_j \cap B(x_k, 45r_k) \neq \emptyset \text{ for some } k \in K_1\}$. Also define $J_2 \equiv (\mathbb{N} \setminus J_1)$ and $K_2 \equiv (\mathbb{N} \setminus K_1)$. We remark that if $k \in K_2$, then there exists $j \in J_2$ such that $B_j \cap B(x_k, 45r_k) \neq \emptyset$ and $135R_j > r_k$. Notice that $B_j \subset \Omega \subset (\bigcup_{k=1}^{\infty} B(x_k, r_k))$. Hence, for each $j \in J_2$, we have $B_j \subset (\bigcup_{k \in K_2, B(x_k, r_k) \cap B_j \neq \emptyset} B(x_k, r_k))$, and then, by $d \leq n$ and the monotonicity of $l^{\frac{d}{n}}$, we see that

$$\begin{aligned} \sum_{k \in K_2} r_k^d &\sim \sum_{k \in K_2} |B(x_k, r_k)|^{\frac{d}{n}} \gtrsim \sum_{j \in J^* \cap J_2} \sum_{k \in K_2, B_j \cap B(x_k, 45r_k) \neq \emptyset} |B(x_k, r_k)|^{\frac{d}{n}} \\ &\gtrsim \sum_{j \in J^* \cap J_2} \left(\sum_{k \in K_2, B_j \cap B(x_k, 45r_k) \neq \emptyset} |B(x_k, r_k)| \right)^{\frac{d}{n}} \gtrsim \sum_{j \in J^* \cap J_2} R_j^d, \end{aligned}$$

which further yields that

$$\sum_{k \in K_1} r_k^d + \sum_{j \in J^* \cap J_2} R_j^d \lesssim \sum_{k \in K} r_k^d.$$

On the other hand, we have

$$\begin{aligned} \Omega \subset \bigcup_{j=1}^{\infty} B_j &\subset \bigcup_{j \in J^*} (15B_j) = \left\{ \bigcup_{j \in J^* \cap J_1} (15B_j) \right\} \cup \left\{ \bigcup_{j \in J^* \cap J_2} (15B_j) \right\} \\ &\subset \left\{ \bigcup_{k \in K_1} B(x_k, 46r_k) \right\} \cup \left\{ \bigcup_{j \in J^* \cap J_2} (15B_j) \right\}. \end{aligned}$$

Notice that for $k \in K_1$, $B(x_k, 45r_k)$ meets B_j for some $j \in \mathbb{N}$ gives us $r_k \geq 135R_j$, which further implies that $B(x_k, 46r_k) \supset B_j$. Also, for $j \in J^*$ and $j' \in \mathbb{N}$, if $B_j \cap B_{j'} \neq \emptyset$, then $(15B_j) \supset B_{j'}$. As a result, we conclude that $\{B(x_k, 46r_k)\}_{k \in K_1} \cup \{15B_j\}_{j \in J^* \cap J_2}$ is the desired covering of Ω and hence,

$$H^d(\Omega, \{B_j\}_{j=1}^{\infty}) \leq \sum_{k \in K_1} (46r_k)^d + \sum_{j \in J^* \cap J_2} (15R_j)^d \lesssim (46)^d H^d(\Omega),$$

which completes the proof of Lemma 3.1. \square

Applying Lemma 3.1, we have the following conclusion.

Lemma 3.2. *Let $\beta \in [1, \infty)$, $\lambda \in (0, \infty)$ and ω be a nonnegative Borel measurable function on \mathbb{R}_+^{n+1} . Then there exists a positive constant C , independent of β , ω and λ , such that*

$$H^d(\{x \in \mathbb{R}^n : N_{\beta}\omega(x) > \lambda\}) \leq C\beta^d H^d(\{x \in \mathbb{R}^n : N\omega(x) > \lambda\}),$$

where $N_{\beta}\omega(x) \equiv \sup_{|y-x| < \beta t} \omega(y, t)$.

Proof. Observe that

$$\{x \in \mathbb{R}^n : N\omega(x) > \lambda\} = \bigcup_{t \in (0, \infty)} \bigcup_{\substack{y \in \mathbb{R}^n \\ \omega(y, t) > \lambda}} B(y, t)$$

and that

$$\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\} = \bigcup_{t \in (0, \infty)} \bigcup_{\substack{y \in \mathbb{R}^n \\ \omega(y, t) > \lambda}} B(y, \beta t).$$

By the Linderöf covering lemma, there exists a countable subset $\{B_l\}_{l=0}^\infty$ of $\{B(y, t) : t \in (0, \infty), y \in \mathbb{R}^n \text{ satisfying } \omega(y, t) > \lambda\}$ such that $\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\} = \{\cup_{l=0}^\infty (\beta B_l)\}$ and $\{x \in \mathbb{R}^n : N\omega(x) > \lambda\} \supset (\cup_{l=0}^\infty B_l)$. By Lemma 3.1, it suffices to prove that

$$H^d(\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\}, \{\beta B_l\}_{l=0}^\infty) \lesssim \beta^d H^d\left(\bigcup_{l=0}^\infty B_l, \{B_l\}_{l=0}^\infty\right).$$

Let $\{B_k^*\}_{k=0}^\infty$ be a ball covering of $\cup_{l \in \mathbb{N}} B_l$ such that $\sum_{k=0}^\infty r_{B_k^*}^d \leq 2H^d(\cup_{l=0}^\infty B_l, \{B_l\}_{l=0}^\infty)$ and that B_k^* engulfs B_l whenever they intersect, where $r_{B_k^*}$ denotes the radius of B_k^* . Therefore, βB_k^* engulfs βB_l whenever they intersect and $\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\} \subset \{\cup_{k=0}^\infty (\beta B_k^*)\}$. We then have

$$2\beta^d H^d\left(\bigcup_{l=0}^\infty B_l, \{B_l\}_{l=0}^\infty\right) \geq \sum_{l=0}^\infty (\beta r_{B_k^*})^d \geq H^d(\{x \in \mathbb{R}^n : N_\beta \omega(x) > \lambda\}, \{\beta B_l\}_{l=0}^\infty),$$

which completes the proof of Lemma 3.2. \square

As an immediate consequence of Lemma 3.2, we have the following result.

Corollary 3.1. *Let $d \in (0, n]$, $\beta \in [1, \infty)$ and ω be a nonnegative measurable function on \mathbb{R}_+^{n+1} . Define $\omega_\beta(x, t) = \sup_{y \in B(x, \beta t)} \omega(y, t)$. Then there exists a positive constant C such that*

$$\int_{\mathbb{R}^n} N\omega_\beta(x) dH^d(x) \leq C\beta^d \int_{\mathbb{R}^n} N\omega(x) dH^d(x).$$

Now we turn to the proof of Theorem 3.1.

Proof of Theorem 3.1. By similarity, we only consider $f\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Similarly to the proof of [31, Theorem 4.1], without loss of generality, we may assume $s = 0$, since this case implies the general case.

By the Aoki theorem (see [2]), there exists a $\kappa \in (0, 1]$ such that $\|\cdot\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^\kappa$ becomes a norm in $f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)$. Let $t \in f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)$. For $Q \in \mathcal{D}(\mathbb{R}^n)$, we write $A \equiv A_0 + A_1$ with $(A_0 t)_Q \equiv \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) \leq \ell(P)\}} a_{QP} t_P$ and $(A_1 t)_Q \equiv \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(P) < \ell(Q)\}} a_{QP} t_P$. By Definition 3.1, we see that for $Q \in \mathcal{D}(\mathbb{R}^n)$,

$$|(A_0 t)_Q| \lesssim \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(Q) \leq \ell(P)\}} \left(\frac{\ell(Q)}{\ell(P)}\right)^{\frac{n+\varepsilon}{2}} \frac{|t_P|}{(1 + [\ell(P)]^{-1}|x_Q - x_P|)^{n+\varepsilon}}.$$

Thus, we have

$$\|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)} \lesssim \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{i=-\infty}^j \sum_{P \in \mathcal{D}_i(\mathbb{R}^n)} 2^{(i-j)\frac{n+\varepsilon}{2}} \right. \right. \right. \\ \left. \left. \left. \times \frac{|t_P|[\omega(\cdot, 2^{-j})]^{-1}}{(1+2^i|x_Q - x_P|)^{n+\varepsilon}} \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}.$$

Let ω be a nonnegative Borel measurable function satisfying (1.1) and

$$\left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |t_Q|^q [\tilde{\chi}_Q \omega(\cdot, 2^{-j})]^{-q} \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \lesssim \|t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}.$$

Let $A_{0,i}(Q) \equiv \{P \in \mathcal{D}_i(\mathbb{R}^n) : 2^i|x_P - x_Q| \leq \sqrt{n}/2\}$ and $A_{m,i}(Q) \equiv \{P \in \mathcal{D}_i(\mathbb{R}^n) : 2^{m-1}\sqrt{n}/2 < 2^i|x_P - x_Q| \leq 2^m\sqrt{n}/2\}$ for all $i \in \mathbb{Z}$ and $m \in \mathbb{Z}_+$. Define $\omega_m(x, t) \equiv 2^{-mn\tau} \sup_{y \in B(x, \sqrt{n}2^{m+1}t)} \omega(y, t)$ for all $(x, t) \in \mathbb{R}_+^{n+1}$. Then $N\omega_m \lesssim 2^{-mn\tau} N_{\sqrt{n}2^{m+2}}\omega$ and $[\omega_m(x, 2^{-j})]^{-1} \omega(y, 2^{-i}) \lesssim 2^{mn\tau}$ for $m \in \mathbb{Z}_+$, $x \in Q$ with $Q \in \mathcal{D}_j(\mathbb{R}^n)$, $y \in P$ with $P \in A_{m,i}(Q)$ and $i \leq j$. Moreover, using Corollary 3.1, we see that a constant multiple of ω_m also satisfies (1.1). Similarly to the proof of Lemma 2.2, we have that for all $x \in Q$,

$$\sum_{P \in A_{m,i}(Q)} \frac{|t_P|[\omega_m(x, 2^{-j})]^{-1}}{(1+2^i|x_Q - x_P|)^{n+\varepsilon}} \lesssim 2^{-m\varepsilon+mn\tau} \text{HL} \left(\sum_{P \in A_{m,i}(Q)} |t_P| \chi_P [\omega(\cdot, 2^{-i})]^{-1} \right) (x).$$

Hence, choosing $\varepsilon > n\tau$, by Fefferman-Stein's vector valued inequality, we obtain

$$\begin{aligned} \|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^\kappa &\lesssim \sum_{m=0}^{\infty} \left\{ \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{i=-\infty}^j \sum_{P \in A_{m,i}(Q)} 2^{(i-j)\frac{n+\varepsilon}{2}} \right. \right. \right. \right. \\ &\quad \left. \left. \left. \times \frac{|t_P|[\omega(\cdot, 2^{-j})]^{-1}}{(1+2^i|x_Q - x_P|)^{n+\varepsilon}} \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} \right\}^\kappa \\ &\lesssim \sum_{m=0}^{\infty} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{i=-\infty}^j \sum_{P \in A_{m,i}(Q)} 2^{(i-j)\frac{n+\varepsilon}{2}} \right. \right. \right. \\ &\quad \left. \left. \left. \times \frac{|t_P|[\omega_m(\cdot, 2^{-j})]^{-1}}{(1+2^i|x_Q - x_P|)^{n+\varepsilon}} \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}^\kappa \\ &\lesssim \sum_{m=0}^{\infty} 2^{m(n\tau-\varepsilon)\kappa} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \chi_Q \left[\sum_{i=-\infty}^j 2^{(i-j)\varepsilon/2} \right. \right. \right. \\ &\quad \left. \left. \left. \times \text{HL} \left(\sum_{P \in A_{m,i}(Q)} |t_P| \tilde{\chi}_P [\omega(\cdot, 2^{-i})]^{-1} \right) \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}^\kappa \lesssim \|t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^\kappa. \end{aligned}$$

The proof for $A_1 t$ is similar. Indeed, we have

$$|(A_1 t)_Q| \lesssim \sum_{\{P \in \mathcal{D}(\mathbb{R}^n) : \ell(P) \leq \ell(Q)\}} \left(\frac{\ell(P)}{\ell(Q)} \right)^{\frac{n+\varepsilon}{2}} \frac{|t_P|}{(1 + [\ell(Q)]^{-1} |x_Q - x_P|)^{n+\varepsilon}}.$$

Thus,

$$\begin{aligned} \|A_1 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)} &\lesssim \inf_{\omega} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{l=0}^{\infty} \sum_{P \in \mathcal{D}_{j+l}(\mathbb{R}^n)} 2^{-l\frac{n+\varepsilon}{2}} \right. \right. \right. \\ &\quad \times \left. \left. \frac{|t_P| [\omega(\cdot, 2^{-j})]^{-1}}{(1 + 2^j |x_Q - x_P|)^{n+\varepsilon}} \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}. \end{aligned}$$

Let $\tilde{A}_{0,j,l}(Q) \equiv \{P \in \mathcal{D}_{j+l}(\mathbb{R}^n) : 2^j |x_P - x_Q| \leq \sqrt{n}/2\}$ and $\tilde{A}_{m,j,l}(Q) \equiv \{P \in \mathcal{D}_{j+l}(\mathbb{R}^n) : 2^{m-1} \sqrt{n}/2 < 2^j |x_P - x_Q| \leq 2^m \sqrt{n}/2\}$ for all $j \in \mathbb{Z}$ and $m, l \in \mathbb{Z}_+$. Set

$$\tilde{\omega}_m(x, s) \equiv 2^{-(m+l)n\tau} \sup\{\omega(y, s) : y \in \mathbb{R}^n, |y - x| < \sqrt{n} 2^{m+l+1} s\}$$

for all $m \in \mathbb{Z}_+$ and $(x, s) \in \mathbb{R}_+^{n+1}$. Similarly, we have that a constant multiple of $\tilde{\omega}_m$ satisfies (1.1) and $[\tilde{\omega}_m(x, 2^{-j})]^{-1} \omega(y, 2^{-j-l}) \lesssim 2^{(m+l)n\tau}$ for $m, l \in \mathbb{Z}_+$, $x \in Q$ with $Q \in \mathcal{D}_j(\mathbb{R}^n)$, $y \in P$ with $P \in \tilde{A}_{m,j,l}(Q)$. Similarly to the proof of Lemma 2.2 again, we see that for all $x \in Q$,

$$\sum_{P \in \tilde{A}_{m,j,l}(Q)} \frac{|t_P| [\tilde{\omega}_m(x, 2^{-j})]^{-1}}{(1 + 2^j |x_Q - x_P|)^{n+\varepsilon}} \lesssim 2^{-m\varepsilon + ln + (m+l)n\tau} \text{HL} \left(\sum_{P \in \tilde{A}_{m,j,l}(Q)} \frac{|t_P| \chi_P}{\omega(\cdot, 2^{-i})} \right) (x).$$

Hence, choosing $\varepsilon > 2n\tau$, similarly to the estimate of $\|A_0 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}$, we also have

$$\begin{aligned} \|A_1 t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^{\kappa} &\lesssim \sum_{m=0}^{\infty} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-\frac{q}{2}} \chi_Q \left[\sum_{l=0}^{\infty} \sum_{P \in \tilde{A}_{m,j,i}(Q)} 2^{-l\frac{n+\varepsilon}{2}} \right. \right. \right. \\ &\quad \times \left. \left. \frac{|t_P| [\omega_m(\cdot, 2^{-j})]^{-1}}{(1 + 2^j |x_Q - x_P|)^{n+\varepsilon}} \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}^{\kappa} \\ &\lesssim \sum_{m=0}^{\infty} 2^{m(n\tau - \varepsilon)\kappa} \left\| \left\{ \sum_{j \in \mathbb{Z}} \sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} \chi_Q \left[\sum_{l=0}^{\infty} 2^{-l(\varepsilon/2 - n\tau)} \right. \right. \right. \\ &\quad \times \left. \left. \text{HL} \left(\sum_{P \in \tilde{A}_{m,j,i}(Q)} |t_P| \tilde{\chi}_P [\omega(\cdot, 2^{-i})]^{-1} \right) \right]^q \right\}^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)}^{\kappa} \lesssim \|t\|_{f\dot{H}_{p,q}^{0,\tau}(\mathbb{R}^n)}^{\kappa}, \end{aligned}$$

which completes the proof of Theorem 3.1. \square

Remark 3.1. We point out that Theorem 3.1 generalizes the corresponding results of Besov Spaces and Triebel-Lizorkin spaces in [11, 12, 13] when $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$. Moreover, the restriction that $\epsilon > 2n\tau$ in Theorem 3.1 is different from the restriction that $\epsilon > 2n(\tau - 1/p)$ in [31, Theorem 4.1] on the spaces $\dot{B}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\dot{F}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

As applications of Theorem 3.1, we establish the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition 3.2. Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$, $\tau \in [0, \frac{1}{(p\vee q)^\gamma}]$ and $Q \in \mathcal{D}(\mathbb{R}^n)$. Set $N \equiv \max(\lfloor -s + 2n\tau \rfloor, -1)$ and $s^* \equiv s - \lfloor s \rfloor$.

(i) A function m_Q is called a *smooth synthesis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q* , if there exist a $\delta \in (\max\{s^*, (s+n\tau)^*\}, 1]$ and $M > n + 2n\tau$ such that $\int_{\mathbb{R}^n} x^\gamma m_Q(x) dx = 0$ if $|\gamma| \leq N$, $|m_Q(x)| \leq |Q|^{-\frac{1}{2}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-\max(M, M-s)}$,

$$|\partial^\gamma m_Q(x)| \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-M} \text{ if } |\gamma| \leq \lfloor s + 3n\tau \rfloor, \quad (3.1)$$

and

$$\begin{aligned} & |\partial^\gamma m_Q(x) - \partial^\gamma m_Q(y)| \\ & \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta \sup_{|z| \leq |x-y|} (1 + [\ell(Q)]^{-1}|x - z - x_Q|)^{-M} \end{aligned} \quad (3.2)$$

if $|\gamma| = \lfloor s + 3n\tau \rfloor$.

A set $\{m_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth synthesis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if each m_Q is a smooth synthesis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q .

(ii) A function b_Q is called a *smooth analysis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q* , if there exist a $\rho \in ((n-s)^*, 1]$ and $M > n + 2n\tau$ such that $\int_{\mathbb{R}^n} x^\gamma b_Q(x) dx = 0$ if $|\gamma| \leq \lfloor s + 3n\tau \rfloor$, $|b_Q(x)| \leq |Q|^{-\frac{1}{2}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-\max(M, M+s+n\tau)}$,

$$|\partial^\gamma b_Q(x)| \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}} (1 + [\ell(Q)]^{-1}|x - x_Q|)^{-M} \text{ if } |\gamma| \leq N, \quad (3.3)$$

and

$$\begin{aligned} & |\partial^\gamma b_Q(x) - \partial^\gamma b_Q(y)| \\ & \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n} - \frac{\delta}{n}} |x - y|^\delta \sup_{|z| \leq |x-y|} (1 + [\ell(Q)]^{-1}|x - z - x_Q|)^{-M} \text{ if } |\gamma| = N. \end{aligned} \quad (3.4)$$

A set $\{b_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth analysis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if each b_Q is a smooth analysis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q .

We remark that if $s + 3n\tau < 0$, then (3.1) and (3.2) are void; if $N < 0$, then (3.3) and (3.4) are void. By a similar argument to the proof of [12, Corollary B.3] (see also [31, Lemma 4.1]), we have the following conclusion.

Lemma 3.3. Let $p \in (1, \infty)$, $q \in [1, \infty)$, $s \in \mathbb{R}$ and $\tau \in [0, \frac{1}{(p\vee q)^\gamma}]$. Then there exist $\varepsilon_1 > 2n\tau$ and a positive constant C such that for all families $\{m_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of smooth synthesis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and families $\{b_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of smooth analysis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $|\langle m_P, b_Q \rangle_{L^2(\mathbb{R}^n)}| \leq C \omega_{QP}(\varepsilon_1)$.

To formulate the molecular decomposition, the following lemma is indispensable.

Lemma 3.4. *Retain the same assumptions as in Lemma 3.3. Let $f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and Φ be a smooth analysis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near a dyadic cube Q . Then $\langle f, \Phi \rangle$ is well defined. Indeed, let $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ be as in (2.2). Then the series*

$$\langle f, \Phi \rangle \equiv \sum_{j \in \mathbb{Z}} \langle \tilde{\varphi}_j * \psi_j * f, \Phi \rangle = \sum_{P \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_P \rangle \langle \psi_P, \Phi \rangle \quad (3.5)$$

converges absolutely and its value is independent of the choices of φ and ψ .

Proof. The same proof as that of [31, Lemma 4.2] works for the absolute convergence of (3.5). We only need to prove that the value of (3.5) is independent of the choices of φ and ψ . By similarity again, we only consider the spaces $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Let $f \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. We claim that $\sum_{j=0}^{\infty} \tilde{\varphi}_j * \psi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n)$. In fact, similarly to the proof of [30, Lemma 2.2], we have that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$|\varphi_j * \phi(x)| \lesssim \|\varphi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \frac{2^{-jM}}{(1 + |x|)^{n+M}},$$

where $M \in \mathbb{N}$ is determined later. Thus,

$$\sum_{j=0}^{\infty} |\langle \tilde{\varphi}_j * \psi_j * f, \phi \rangle| \lesssim \|\varphi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \sum_{j=0}^{\infty} 2^{-jM} \int_{\mathbb{R}^n} \frac{|\psi_j * f(x)|}{(1 + |x|)^{n+M}} dx.$$

Recall again that $\omega(x, t) \lesssim t^{-n\tau}$ for all nonnegative Borel measurable functions ω on \mathbb{R}_+^{n+1} satisfying (1.1). Letting $M > \max(0, n\tau - s)$, by Hölder's inequality, we then obtain

$$\begin{aligned} \sum_{j=0}^{\infty} |\langle \tilde{\varphi}_j * \psi_j * f, \phi \rangle| &\lesssim \|\varphi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \sum_{j=0}^{\infty} 2^{-jM+jn\tau} \int_{\mathbb{R}^n} \frac{|\psi_j * f(x)| [\omega(x, 2^{-j})]^{-1}}{(1 + |x|)^{n+M}} dx \\ &\lesssim \|\varphi\|_{S_{M+1}} \|\phi\|_{S_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}, \end{aligned}$$

which implies that $\sum_{j=0}^{\infty} \tilde{\varphi}_j * \psi_j * f$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Thus, the claim is true.

We need to handle carefully the remaining summation: $\sum_{j=-\infty}^{-1} \tilde{\varphi}_j * \psi_j * f$. In general it is not possible to prove that $\sum_{j=-\infty}^{-1} \tilde{\varphi}_j * \psi_j * f$ is convergent in $\mathcal{S}'(\mathbb{R}^n)$. Therefore, we pass to its partial derivatives. Choose $\gamma \in \mathbb{Z}_+^n$ such that $|\gamma| > s - n\tau - n/p$. Then using Hölder's inequality, similarly to the previous estimate, we obtain that for all $x \in \mathbb{R}^n$,

$$\begin{aligned} \sum_{j=-\infty}^{-1} |\partial^\gamma (\tilde{\varphi}_j * \psi_j * f)(x)| &\lesssim \sum_{j=-\infty}^{-1} 2^{j(n+|\gamma|)} \|\varphi\|_{S_{M+1}} \int_{\mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1 + 2^j|x-y|)^{n+M+|\gamma|}} dy \\ &\lesssim \sum_{j=-\infty}^{-1} 2^{j(|\gamma|-s+n\tau+\frac{n}{p})} \|\varphi\|_{S_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\ &\lesssim \|\varphi\|_{S_{M+1}} \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}. \end{aligned}$$

Therefore, it follows from the well-known result in [12, Remark B.4] or [5, Lemma 5.4] that there exist a sequence $\{P_N\}_{N \in \mathbb{N}}$ of polynomials on \mathbb{R}^n with degree no more than

$\max(-1, \lfloor s - n\tau - n/p \rfloor)$ and $g \in \mathcal{S}'(\mathbb{R}^n)$ such that $g = \lim_{N \rightarrow \infty} (\sum_{j=-N}^{\infty} \tilde{\varphi}_j * \psi_j * f + P_N)$ in $\mathcal{S}'(\mathbb{R}^n)$ and g is a representative of the equivalence class $f + \mathcal{P}(\mathbb{R}^n)$; see [12, pp. 153-154]. Using [5, Lemma 5.4] and repeating the argument in [12, pp. 153-154], we obtain that the value of (3.5) is independent of the choices of φ and ψ , which completes the proof of Lemma 3.4. \square

With Theorem 3.1, Lemmas 3.3 and 3.4, we now have the following smooth molecular decomposition of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. The proof of Theorem 3.2 parallels the proofs of [31, Theorem 4.2] and [12, Theorems 3.5, 3.7]. We omit the details.

Theorem 3.2. *Let s, p, q and τ be as in Lemma 3.3.*

(i) *If $\{m_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ is a family of smooth synthesis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then there exists a positive constant C such that for all $t = \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,*

$$\left\| \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q m_Q \right\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

(ii) *If $\{b_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ is a family of smooth analysis molecules for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, then there exist a positive constant C such that for all $f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$,*

$$\|\{\langle f, b_Q \rangle\}_{Q \in \mathcal{D}(\mathbb{R}^n)}\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

Theorem 3.2 generalizes the well known results on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$ in [10, 11, 12, 13, 3, 5] by taking $\tau = 0$.

Next we establish the smooth atomic decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Definition 3.3. Let $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$, τ and N be as in Definition 3.2. A function a_Q is called a *smooth atom* for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near a dyadic cube Q , if there exist \tilde{K} and \tilde{N} with $\tilde{K} \geq \max(\lfloor s + 3n\tau + 1 \rfloor, 0)$ and $\tilde{N} \geq N$ such that a_Q satisfies the following support, regularity and moment conditions: $\text{supp } a_Q \subset 3Q$, $\|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}}$ if $|\gamma| \leq \tilde{K}$, and $\int_{\mathbb{R}^n} x^\gamma a_Q(x) dx = 0$ if $|\gamma| \leq \tilde{N}$.

A set $\{a_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of functions is called a family of smooth atoms for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, if each a_Q is a smooth atom for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q .

Remark 3.2. We point out that in Definition 3.3, the regularity condition of smooth atoms can be strengthened into that $\|\partial^\gamma a_Q\|_{L^\infty(\mathbb{R}^n)} \leq |Q|^{-\frac{1}{2} - \frac{|\gamma|}{n}}$ for all $|\gamma| \leq M$, where M can be any sufficiently large constant depending on s, τ, p and q ; see Grafakos [15, Definition 6.6.2] for the details.

It is clear that every smooth atom for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ is a constant multiple of a smooth synthesis molecule $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. Once we establish Theorem 3.2, an argument used in [12, pp. 60-61] or [5, pp. 1495-1497] yields the following conclusion; we omit the details.

Theorem 3.3. *Let s, p, q, τ be as in Lemma 3.3. Then for each $f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, there exist a family $\{a_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of smooth atoms for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, a coefficient sequence $t \equiv \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, and a positive constant C such that $f = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}'(\mathbb{R}^n)$ and $\|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \|f\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.*

Conversely, there exists a positive constant C such that for all families $\{a_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)}$ of smooth atoms for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and coefficient sequences $t \equiv \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \in a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, $\|\sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q a_Q\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \leq C \|t\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$.

Theorem 3.3 again generalizes the well known results on $\dot{B}_{p,q}^s(\mathbb{R}^n)$ and $\dot{F}_{p,q}^s(\mathbb{R}^n)$ in [10, 11, 12, 13] (see also [3, 5, 15]) by taking $\tau = 0$.

4 Pseudo-differential operators and trace theorems

In this section, we give some applications of the smooth atomic and molecular decomposition characterizations of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, including the boundedness of pseudo-differential operators with homogeneous symbols in these spaces and their trace properties. We first recall the notion of homogeneous symbols; see, for example, [16].

Definition 4.1. Let $m \in \mathbb{Z}$. A smooth function a defined on $\mathbb{R}_x^n \times (\mathbb{R}_\xi^n \setminus \{0\})$ belongs to the class $\dot{S}_{1,1}^m(\mathbb{R}^n)$, if a satisfies the following differential inequalities that for all $\alpha, \beta \in \mathbb{Z}_+^n$,

$$\sup_{x \in \mathbb{R}^n, \xi \in (\mathbb{R}^n \setminus \{0\})} |\xi|^{-m-|\alpha|+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| < \infty.$$

As an application of the smooth molecular decomposition of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ (Theorem 3.2) and the Calderón reproducing formula (2.2), we have the following conclusion.

Theorem 4.1. Let $m \in \mathbb{Z}$, $s \in \mathbb{R}$, $p \in (1, \infty)$, $q \in [1, \infty)$ and $\tau \in [0, \frac{1}{(p \vee q)^\tau}]$. Let a be a symbol in $\dot{S}_{1,1}^m(\mathbb{R}^n)$ and $a(x, D)$ be the pseudo-differential operator such that

$$a(x, D)f(x) \equiv \int_{\mathbb{R}^n} a(x, \xi)(\mathcal{F}f)(\xi)e^{ix\xi} d\xi$$

for all smooth synthesis molecules for $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Assume that its formal adjoint $a(x, D)^*$ satisfies $a(x, D)^*(x^\beta) = 0$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ for all $\beta \in \mathbb{Z}_+^n$ with $|\beta| \leq \max\{-s + 2n\tau, -1\}$. Then $a(x, D)$ is a bounded linear operator from $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ to $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

Proof. The proof is similar to that in [14, 9, 23, 24, 16]; see also [22]. We abbreviate $T \equiv a(x, D)$ for simplicity. Let $f \in A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ and φ be as in Definition 1.1 such that for all $\xi \in \mathbb{R}^n$, $\sum_{j \in \mathbb{Z}} |\mathcal{F}\varphi(2^{-j}\xi)|^2 = \chi_{\mathbb{R}^n \setminus \{0\}}(\xi)$. Then by the Calderón reproducing formula (2.2), we have $f \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_Q \rangle \varphi_Q$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$; moreover, by the φ -transform characterization of $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ (see Theorem 2.1), we see that $\|\{\langle f, \varphi_Q \rangle\}_{Q \in \mathcal{D}(\mathbb{R}^n)}\|_{a\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$, or equivalently, $\|\{|Q|^{-\frac{m}{n}} \langle f, \varphi_Q \rangle\}_{Q \in \mathcal{D}(\mathbb{R}^n)}\|_{a\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$.

We claim that $T(f) \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \langle f, \varphi_Q \rangle T(\varphi_Q)$ in $\mathcal{S}'_\infty(\mathbb{R}^n)$ with $\|T(f)\|_{A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)}$. To this end, by Theorem 3.2 (i), it suffices to show that every $|Q|^{\frac{m}{n}} T(\varphi_Q)$ is a constant multiple of a synthesis molecule for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near Q . This fact was established by Grafakos and Torres [16]; see also [22]. We then conclude that T is bounded from $A\dot{H}_{p,q}^{s+m,\tau}(\mathbb{R}^n)$ to $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, which completes the proof of Theorem 4.1. \square

We remark that Theorem 4.1 generalizes the corresponding classical results in Besov spaces and Triebel-Lizorkin spaces obtained by Grafakos and Torres [16, Theorems 1.1 and 1.2] when $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$.

As an application of smooth atomic decomposition of $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, we are now going to show the trace theorem. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set $x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Theorem 4.2. *Let $n \geq 2$, $p \in (1, \infty)$, $q \in [1, \infty)$, $\tau \in [0, \frac{n-1}{n(p \vee q)'}]$ and $s \in (\frac{1}{p} + 2n\tau, \infty)$. Then there exists a surjective and continuous operator*

$$\text{Tr} : f \in A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n) \mapsto \text{Tr}(f) \in A\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$$

such that $\text{Tr}(f)(x') = f(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$ and smooth atoms f for $A\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$.

To prove this theorem, we need the following technical lemma.

Lemma 4.1. *Let $d \in (0, n]$ and Ω be an open set in \mathbb{R}^n . Define*

$$H_*^d(\Omega) \equiv \inf \left\{ \sum_{j=1}^{\infty} r_j^d : \Omega \subset \bigcup_{j=1}^{\infty} B(x_j, r_j), r_j > \frac{\text{dist}(x_j, \partial\Omega)}{10000} \right\}.$$

Then $H^d(\Omega)$ and $H_*^d(\Omega)$ are equivalent for all Ω .

Proof. The inequality $H^d(\Omega) \leq H_*^d(\Omega)$ is trivial from the definitions. To prove the converse, we choose a ball covering $\{B(x_j, r_j)\}_{j=1}^{\infty}$ of Ω such that $\sum_{j=1}^{\infty} r_j^d \leq 2H^d(\Omega)$. Let $\{B(X_j, R_j)\}_{j=1}^{\infty}$ be a Whitney covering of Ω satisfying $\Omega = \bigcup_{j=1}^{\infty} B(X_j, R_j)$, $R_j/1000 \leq \text{dist}(X_j, \partial\Omega) \leq R_j/100$ and $\sum_{j \in \mathbb{N}} \chi_{R_j} \leq C_n$; see, for example, [15, Proposition 7.3.4]. Set

$$J_1 \equiv \{j \in \mathbb{N} : (B(X_j, R_j) \cap B(x_k, r_k)) \neq \emptyset \text{ and } R_j \leq 4r_k \text{ for some } k \in \mathbb{N}\}$$

and $J_2 \equiv (\mathbb{N} \setminus J_1)$. Notice that if $k \in \mathbb{N}$ satisfies $(B(X_j, R_j) \cap B(x_k, r_k)) \neq \emptyset$ for some $j \in J_2$, then $B(x_k, r_k) \subset B(X_j, 2R_j)$, since $r_k < R_j/4$. With this in mind, we define

$$K_2 \equiv \{k \in \mathbb{N} : (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset \text{ for some } j \in J_2\},$$

and $K_1 \equiv (\mathbb{N} \setminus K_2)$. It is easy to see that

$$\bigcup_{k=1}^{\infty} B(x_k, r_k) \subset \left(\bigcup_{k \in K_1} B(x_k, r_k) \bigcup \bigcup_{j \in J_2} B(X_j, 2R_j) \right). \quad (4.1)$$

Furthermore, for each $k \in \mathbb{N}$, the cardinality of the set $\{j \in J_2 : (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset\}$ is bounded by a constant depending only on the dimension. Hence, we have

$$\begin{aligned} \sum_{k=1}^{\infty} r_k^d &= \sum_{k \in K_1} r_k^d + \sum_{k \in K_2} r_k^d \sim \sum_{k \in K_1} r_k^d + \sum_{j \in J_2} \left(\sum_{k \in K_2, (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset} r_k^d \right) \\ &\sim \sum_{k \in K_1} r_k^d + \sum_{j \in J_2} \left(\sum_{k \in K_2, (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset} |B(x_k, r_k)|^{\frac{d}{n}} \right). \end{aligned}$$

Notice that $B(X_j, R_j) \subset \Omega \subset (\cup_{k=1}^{\infty} B(x_k, r_k))$. Then for each $j \in J_2$, we have

$$B(X_j, R_j) \subset \left\{ \bigcup_{k \in K_2, (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset} B(x_k, r_k) \right\}$$

Since $d \in (0, n]$, by the monotonicity of $l_n^{\frac{d}{n}}$, we see that

$$\begin{aligned} & \left(\sum_{k \in K_2, (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset} |B(x_k, r_k)|^{\frac{d}{n}} \right) \\ & \geq \left(\sum_{k \in K_2, (B(x_k, r_k) \cap B(X_j, R_j)) \neq \emptyset} |B(x_k, r_k)| \right)^{\frac{d}{n}} \geq |B(X_j, R_j)|^{\frac{d}{n}}. \end{aligned}$$

As a consequence, $\sum_{k=0}^{\infty} r_k^d \gtrsim \sum_{k \in K_1} r_k^d + \sum_{j \in J_2} R_j^d$, which combined with (4.1) yields that $H_*^d(\Omega) \leq \sum_{k \in K_1} r_k^d + \sum_{j \in J_2} (2R_j)^d \lesssim \sum_{k \in K_1} r_k^d + \sum_{j \in J_2} R_j^d \lesssim \sum_{k=0}^{\infty} r_k^d \lesssim H^d(\Omega)$. This finishes the proof of Lemma 4.1. \square

Proof of Theorem 4.2. For similarity, we concentrate on the space $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$. By Theorem 3.3, any $f \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ admits a smooth atomic decomposition $f = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q a_Q$ in $\mathcal{S}'_{\infty}(\mathbb{R}^n)$, where each a_Q is a smooth atom for $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $t \equiv \{t_Q\}_{Q \in \mathcal{D}(\mathbb{R}^n)} \subset \mathbb{C}$ satisfies $\|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}$. Since $s > 1/p + 2n\tau$, there is no need to postulate any moment condition on a_Q . Define

$$\text{Tr}(f)(*)' \equiv \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} t_Q a_Q(*', 0) = \sum_{Q \in \mathcal{D}(\mathbb{R}^n)} \frac{t_Q}{[\ell(Q)]^{\frac{1}{2}}} [\ell(Q)]^{\frac{1}{2}} a_Q(*', 0).$$

By the support condition of smooth atoms, the above summation can be re-written as

$$\text{Tr}(f)(*)' \equiv \sum_{i=0}^2 \sum_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \frac{t_{Q' \times [(i-1)\ell(Q'), i\ell(Q')]} [\ell(Q')]^{\frac{1}{2}} a_{Q' \times [(i-1)\ell(Q'), i\ell(Q')]}(*', 0)}{[\ell(Q')]^{\frac{1}{2}}}. \quad (4.2)$$

We need to show that (4.2) converges in $\mathcal{S}'_{\infty}(\mathbb{R}^{n-1})$ and

$$\|\text{Tr}(f)\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

To this end, by Theorem 3.3, it suffices to prove that each $[\ell(Q')]^{\frac{1}{2}} a_{Q' \times [(i-1)\ell(Q'), i\ell(Q')]}(*', 0)$ is a smooth atom for $B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$ supported near Q' and for all $i \in \{0, 1, 2\}$,

$$\left\| \left\{ [\ell(Q')]^{-\frac{1}{2}} t_{Q' \times [(i-1)\ell(Q'), i\ell(Q')]} \right\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \right\|_{b\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} < \infty. \quad (4.3)$$

Indeed, it was already proved in [22] that $[\ell(Q')]^{\frac{1}{2}} a_{Q' \times [(i-1)\ell(Q'), i\ell(Q')]}(*', 0)$ is a smooth atom for $B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$. By similarity, we only prove (4.3) when $i = 1$. Let ω be a

nonnegative function on \mathbb{R}_+^{n+1} satisfying (1.1) and

$$\left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q \in \mathcal{D}_j(\mathbb{R}^n)} |Q|^{-(\frac{s}{n} + \frac{1}{2})p} |t_Q|^p \int_Q [\omega(x, 2^{-j})]^{-p} dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}.$$

For all $\lambda \in (0, \infty)$, set $E_\lambda \equiv \{x \in \mathbb{R}^n : [N\omega(x)]^{(p \vee q)'} > \lambda\}$. Then there exists a ball covering $\{B_m\}_m$ of E_λ such that

$$H^{n\tau(p \vee q)'}(E_\lambda) \sim \sum_m r_{B_m}^{n\tau(p \vee q)'}, \quad (4.4)$$

where r_{B_m} denotes the radius of B_m . Let $\tilde{H}^{n\tau(p \vee q)'}$ be the $(n-1)\frac{n\tau}{n-1}(p \vee q)'$ -Hausdorff capacity in \mathbb{R}^{n-1} and define $\tilde{\omega}$ on \mathbb{R}_+^n by setting, for all $x' \in \mathbb{R}^{n-1}$ and $t \in (0, \infty)$, $\tilde{\omega}(x', t) \equiv \tilde{C} \sup_{\{x_n \in \mathbb{R} : |x_n| < t\}} \omega((x', x_n), t)$, where \tilde{C} is a positive constant chosen so that $N\tilde{\omega}(x') \leq N\omega(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$. Therefore, if $[N\tilde{\omega}(x')]^{(p \vee q)' > \lambda}$, then $[N\omega(x', 0)]^{(p \vee q)' > \lambda}$, and hence $(x', 0) \in B_m$ for some m , which further implies that $\tilde{E}_\lambda \equiv \{x' \in \mathbb{R}^{n-1} : [N\tilde{\omega}(x')]^{(p \vee q)' > \lambda\} \subset (\cup_m B_m^*)$, where B_m^* is the projection of B_m from \mathbb{R}^n to \mathbb{R}^{n-1} . This combined with (4.4) further yields that

$$\int_{\mathbb{R}^{n-1}} [N\tilde{\omega}(x')]^{(p \vee q)'} d\tilde{H}^{n\tau(p \vee q)'}(x') = \int_0^\infty \tilde{H}^{n\tau(p \vee q)'}(\tilde{E}_\lambda) d\lambda \lesssim \int_0^\infty H^{n\tau(p \vee q)'}(E_\lambda) d\lambda \lesssim 1.$$

Furthermore,

$$\begin{aligned} & \left\| \left\{ [\ell(Q')]^{-\frac{1}{2}} t_{Q' \times [0, \ell(Q')]} \right\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \right\|_{b\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \\ & \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} [\ell(Q')]^{-sp - \frac{np}{2} + 1} |t_{Q' \times [0, \ell(Q')]}|^p \int_{Q'} [\tilde{\omega}(x', 2^{-j})]^{-p} dx' \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ & \lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} [\ell(Q')]^{-sp - \frac{np}{2}} |t_{Q' \times [0, \ell(Q')]}|^p \int_Q [\omega(x, 2^{-j})]^{-p} dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ & \lesssim \|t\|_{b\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)}, \end{aligned}$$

which implies that Tr is well defined and bounded from $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ to $B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$.

Let us show that Tr is surjective. To this end, for any $f \in B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$, by Theorem 3.3, there exist smooth atoms $\{a_{Q'}\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})}$ for $B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})$ and coefficients $t \equiv \{t_{Q'}\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})}$ such that $f = \sum_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} t_{Q'} a_{Q'}$ in $\mathcal{S}'_\infty(\mathbb{R}^{n-1})$ and $\|t\|_{b\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}$. Let $\varphi \in C_c^\infty(\mathbb{R})$ with $\text{supp } \varphi \subset (-\frac{1}{2}, \frac{1}{2})$ and $\varphi(0) = 1$. For all $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$ and $x \in \mathbb{R}$, set $\varphi_{Q'}(x) \equiv \varphi(2^{-\log_2 \ell(Q')} x)$. Under this notation, we define $F \equiv \sum_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} t_{Q'} a_{Q'} \otimes \varphi_{Q'}$. It is easy to check that for all $Q' \in \mathcal{D}(\mathbb{R}^{n-1})$,

$[\ell(Q')]^{-\frac{1}{2}} a_{Q'} \otimes \varphi_{Q'}$ is a smooth atom for $B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ supported near $Q' \times [0, \ell(Q'))$. Hence, to show $F \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$, by Theorem 3.3, it suffices to prove that

$$\left\| \{[\ell(Q')]^{\frac{1}{2}} t_{Q'}\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \right\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}.$$

Let $\tilde{\omega}$ satisfy $\int_{\mathbb{R}^{n-1}} [N\tilde{\omega}(x')]^{(p\vee q)'} d\tilde{H}^{n\tau(p\vee q)'}(x') \leq 1$ and

$$\left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} |Q'|^{-(\frac{s-1/p}{n-1} + \frac{1}{2})p} |t_{Q'}|^p \int_{Q'} [\tilde{\omega}(x', 2^{-j})]^{-p} dx' \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \lesssim \|t\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}.$$

By Lemma 4.1, for each $\lambda \in (0, \infty)$, there exists a ball covering $\{B_m^*\}_m \equiv \{B(x_{B_m^*}, r_{B_m^*})\}_m$ of $\tilde{E}_\lambda \equiv \{x' \in \mathbb{R}^{n-1} : [N\tilde{\omega}(x')]^{(p\vee q)'} > \lambda\}$ such that $\sum_m r_{B_m^*}^{n\tau(p\vee q)'} \sim \tilde{H}_*^{n\tau(p\vee q)'}(\tilde{E}_\lambda) \sim \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda)$ and that $r_{B_m^*} > \text{dist}(x_{B_m^*}, \partial\tilde{E}_\lambda)/10000$ for all m . For all $x = (x', x_n) \in \mathbb{R}^n$ and $t \in (0, \infty)$, define $\omega(x, t) \equiv \tilde{\omega}(x', t)\chi_{[0,t)}(x_n)$. Notice that if $N\omega(x', x_n) > \lambda^{\frac{1}{(p\vee q)'}}$, then $\tilde{\omega}(y', t) = \omega((y', y_n), t) > \lambda^{\frac{1}{(p\vee q)'}}$ for some $|(y', y_n) - (x', x_n)| < t$ and $y_n \in [0, t)$. Then $N\tilde{\omega}(y') > \lambda^{\frac{1}{(p\vee q)'}}$ and thus, $y' \in B_m^*$ for some m . Since for all $z' \in B(y', t)$, $N\tilde{\omega}(z') \geq \tilde{\omega}(y', t) > \lambda^{\frac{1}{(p\vee q)'}}$, we see that $B(y', t) \subset \tilde{E}_\lambda \subset (\cup_m B_m^*)$, and hence, $t \leq 10000r_{B_m^*}$. Notice that $x_n \in [0, t)$. We have $(x', x_n) \in (20000B_m^*) \times [0, 20000r_{B_m^*})$ and $E_\lambda \subset \cup_m (20000B_m^*) \times [0, 20000r_{B_m^*})$, which further implies that $H^{n\tau(p\vee q)'}(E_\lambda) \lesssim \sum_m r_{B_m^*}^{n\tau(p\vee q)'} \lesssim \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda)$ and

$$\begin{aligned} \int_{\mathbb{R}^n} [N\omega(x', x_n)]^{(p\vee q)'} dH^{n\tau(p\vee q)'}(x) &= \int_0^\infty H^{n\tau(p\vee q)'}(E_\lambda) d\lambda \lesssim \int_0^\infty \tilde{H}^{n\tau(p\vee q)'}(\tilde{E}_\lambda) d\lambda \\ &\lesssim \int_{\mathbb{R}^{n-1}} [N\tilde{\omega}(x')]^{(p\vee q)'} d\tilde{H}^{n\tau(p\vee q)'}(x') \lesssim 1. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\left\| \{[\ell(Q')]^{\frac{1}{2}} t_{Q'}\}_{Q' \in \mathcal{D}(\mathbb{R}^{n-1})} \right\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \\ &\lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} [\ell(Q')]^{-(\frac{s}{n} + \frac{1}{2})pn + \frac{p}{2}} |t_{Q'}|^p \int_{Q' \times [0, \ell(Q'))} [\omega(x, 2^{-j})]^{-p} dx \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\lesssim \left\{ \sum_{j \in \mathbb{Z}} \left[\sum_{Q' \in \mathcal{D}_j(\mathbb{R}^{n-1})} |Q'|^{-(\frac{s-1/p}{n-1} + \frac{1}{2})p} |t_{Q'}|^p \int_{Q'} [\tilde{\omega}(x', 2^{-j})]^{-p} dx' \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} \\ &\lesssim \|t\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}, \end{aligned}$$

which implies that $F \in B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)$ and $\|F\|_{B\dot{H}_{p,q}^{s,\tau}(\mathbb{R}^n)} \lesssim \|f\|_{B\dot{H}_{p,q}^{s-\frac{1}{p}, \frac{n}{n-1}\tau}(\mathbb{R}^{n-1})}$. Furthermore, the definition of F implies $\text{Tr}(F) = f$, which completes the proof of Theorem 4.2. \square

We point out that Theorem 4.2 generalizes the corresponding classical results on Besov and Triebel-Lizorkin spaces for $p \in (1, \infty)$ and $q \in [1, \infty)$ by taking $\tau = 0$; see, for example, Jawerth [17, Theorem 5.1], [18, Theorem 2.1] and Frazier-Jawerth [12, Theorem 11.1].

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